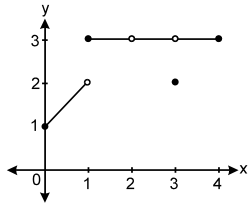
SOLUTIONS OF LIMITS, CONTINUITY & DERIVABILITY

EXERCISE # 1

PART–1

Section (A)

A-1. 

By checking limits

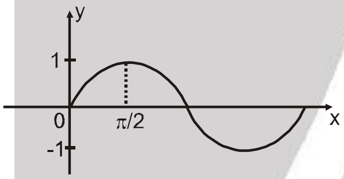
(i) $\lim\limits_{x\to1} f(x)$ = does not exist since L.H.L. $\neq$ R.H.L. **(ii)** $\lim\limits_{x\to2} f(x) = 3$

(iii) $\lim\limits_{x\to3} f(x) = 3$ **(iv)** $\lim\limits_{x\to1.99} f(x) = 3$ **(v)** $\lim\limits_{x\to3^-} f(x) = 3$

A-2. **(i)** $\lim\limits_{x\to2} (x + \sin x) = 2 + \sin 2$ **(ii)** $\lim\limits_{x\to3} \tan x - 2^x = \tan 3 - 8$

(iii) $\lim\limits_{x\to\frac{3}{4}} x \cos x = \frac{3}{4} \cos \frac{3}{4}$ **(iv)** $\lim\limits_{x\to5} x^x = 5^5$

(v) $\lim\limits_{x\to1} \frac{e^x}{\sin x} = \frac{e}{\sin 1}$

A-3. **(i)** 

$\lim\limits_{x\to\frac{\pi}{2}} [\sin x]$ By graph of sin x

R.H.L = $\lim\limits_{x\to\frac{\pi^+}{2}} [\sin x] = 0$

L.H.L. = $\lim\limits_{x\to\frac{\pi^-}{2}} [\sin x] = 0$ so $\lim\limits_{x\to\frac{\pi}{2}} [\sin x] = 0$

(ii) $\lim\limits_{x\to2} \left\{\frac{x}{2}\right\}$

R.H.L. = $\lim\limits_{h\to0} \left\{\frac{2+h}{2}\right\} = \lim\limits_{h\to0} \left\{1+\frac{h}{2}\right\} = \lim\limits_{h\to0} \left\{\frac{h}{2}\right\} = \lim\limits_{h\to0} \frac{h}{2} = 0$

L.H.L. = $\lim\limits_{h\to0} \left\{\frac{2-h}{2}\right\} = \lim\limits_{h\to0} \left\{1-\frac{h}{2}\right\} = \lim\limits_{h\to0} \left(\frac{1-h}{2}\right) = 1$

L.H.L. $\neq$ R.H.L. so $\lim\limits_{x\to0} \left\{\frac{x}{2}\right\}$ does not exist.

(iii) $\lim\limits_{x\to\pi} \text{sgn}\,[\tan x]$

L.H.L. = $\lim\limits_{x\to\pi^-} \text{sgn}\,[\tan x] = \lim\limits_{h\to0} \text{sgn}\,[\tan(\pi - h)] = \lim\limits_{h\to0} \text{sgn}\,(-\text{ve}) = -1$

R.H.L. = $\lim\limits_{x\to\pi^-} \text{sgn}\,[\tan x] = \lim\limits_{h\to0} \text{sgn}\,[\tan(\pi + h)] = \lim\limits_{h\to0} \text{sgn}\,(+\text{ve}) = 0$

L.H.L. $\neq$ R.H.L.
so limit does not exist

(iv) $\lim\limits_{x\to1} \sin^{-1}(\ell n\, x) = \sin^{-1}\left(\lim\limits_{x\to1} \ell n x\right) = \sin^{-1}(0) = 0$ so $\lim\limits_{x\to1} \sin^{-1}(\ell n\, x) = 0$

A-4. **(i)** If $f(x) = \begin{cases} x+1 & , \ x<1 \\ 2x-3 & , \ x \geq 1 \end{cases} \Rightarrow \lim\limits_{x\to 1} f(x)$ exist if L.H.L. = R.H.L.

R.H.L. = $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (2x-3) = \lim\limits_{h\to 0} [2(1+h)-3] = -1$

L.H.L. $= \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (x+1) = 2$

L.H.L. $\neq$ R.H.L.

so limit does not exist at x = 1

(ii) $\lim\limits_{x\to 1} f(x)$ exist if R.H.L. = L.H.L.

so $\lim\limits_{x\to 1^+} (2x-3) = \lim\limits_{x\to 1^-} (x+\lambda) \Rightarrow 2-3 = 1+\lambda \Rightarrow \lambda = -2$

A-5. $f[g(x)] = \begin{cases} [g(x)]^2+2 & , \ x \geq 2 \\ 1-g(x) & , \ x < 2 \end{cases} = \begin{cases} (2x)^2+2 & ; \ 2x \geq 2 \text{ and } x>1 \\ (3-x)^2+2 & ; \ 3-x \geq 2 \text{ and } x \leq 1 \\ 1-2x & ; \ 2x < 2 \text{ and } x > 1 \\ 1-(3-x) & ; \ 3-x<2 \text{ and } x \leq 1 \end{cases}$

$= \begin{cases} 4x^2+2 & ; \ x \geq 1 \text{ and } x>1 \\ x^2-6x+11 & ; \ x \leq 1 \text{ and } x \leq 1 \\ 1-2x & ; \ x < 1 \text{ and } x > 1 \\ x-2 & ; \ x>1 \text{ and } x \leq 1 \end{cases}$

$f[g(x)] = \begin{cases} 4x^2+2 & , \ x>1 \\ x^2-6x+11 & , \ x \leq 1 \end{cases}$

so $\lim\limits_{x\to 1} f[g(x)]$

R.H.L.= $\lim\limits_{x\to 1^+} (4x^2+2) = 6$

L.H.L.= $\lim\limits_{x\to 1^-} (x^2-6x+11) = 6$

L.H.L.= R.H.L.

so $\lim\limits_{x\to 1} f[g(x)] = 6$

A-6. **(i)** $\lim\limits_{x\to 0^+} \dfrac{[x]}{x} = \left(\dfrac{0}{+\text{ve value}}\right) \to$ Not an indeterminate form

(ii) $\lim\limits_{x\to -\infty} \sqrt{x^2+1} - x = \infty + \infty = \infty \to$ Not an indeterminate form

(iii) $\lim\limits_{x\to \frac{\pi}{2}} (\tan x)^{\tan 2x} = (\infty)^0$ form $\to$ Yes

(iv) $\lim\limits_{x\to 1^+} \left(\{x\}\right)^{\frac{1}{\ell nx}}$; $\lim\limits_{h\to 0} \{1+h\}^{\frac{1}{\ell n(1+h)}} \lim\limits_{h\to 0} \{h\}^{\frac{1}{\ell n(1+h)}} = (0^\infty \text{ form}) = 0 \to$ Not an indeterminate form

Section (B)

B-1. **(i)** $\lim\limits_{x\to -1} \dfrac{x^3-3x+1}{x-1} = \dfrac{-1+3+1}{-2} = \dfrac{-3}{2}$

(ii) $\lim\limits_{x\to 1} \dfrac{4x^3-x^2+2x-5}{x^6+5x^3-2x-4} \left(\dfrac{0}{0}\text{form}\right) = \lim\limits_{x\to 1} \dfrac{(x-1)\ (4x^2+3x+5)}{(x-1)(x^5+x^4+x^3+6x^2+6x+4)}$

$= \dfrac{4+3+5}{1+1+1+6+6+4} = \dfrac{12}{19}$

(iii) $\lim\limits_{x\to a} \dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$, $a \neq 0$ $\left(\dfrac{0}{0}\text{form}\right)$

$\lim\limits_{x\to a} \dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} \times \dfrac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{3a+x}+2\sqrt{x}} \times \dfrac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}} = \lim\limits_{x\to a} \dfrac{(a+2x-3x)}{(3a+x)-4x} \times \dfrac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}}$

$= \lim\limits_{x\to a} \dfrac{a-x}{3(a-x)} \times \dfrac{2\sqrt{a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}} = \dfrac{1}{3}\left[\dfrac{2}{\sqrt{3}}\right] = \dfrac{2}{3\sqrt{3}}$

B-2. **(i)** $\lim\limits_{x\to 0} \dfrac{1-\cos 4x}{1-\cos 5x}\left(\dfrac{0}{0}\text{form}\right) = \lim\limits_{x\to 0} \dfrac{2\sin^2 2x}{2\sin^2 \dfrac{5x}{2}} = \lim\limits_{x\to 0} \dfrac{2^2\left(\dfrac{\sin\ 2x}{2x}\right)^2}{\left(\dfrac{5}{2}\right)^2\left(\dfrac{\sin\dfrac{5x}{2}}{\dfrac{5x}{2}}\right)^2} = \dfrac{16}{25}$

(ii) $\lim\limits_{x\to \frac{\pi}{6}} \dfrac{\sqrt{3}\sin x - \cos x}{x-\dfrac{\pi}{6}}\left(\dfrac{0}{0}\text{form}\right)$ using L' Hospital rule $= \lim\limits_{x\to \frac{\pi}{6}} \dfrac{\sqrt{3}\cos x + \sin x}{1} = \dfrac{3}{2}+\dfrac{1}{2} = 2$

(iii) $\lim\limits_{x\to 0} \dfrac{\tan 3x - 2x}{3x-\sin^2 x}\ \lim\limits_{x\to 0} \dfrac{3\ .\ \dfrac{\tan\ 3x}{3x} - 2}{3-\sin\ x.\ \dfrac{\sin x}{x}} = \dfrac{3(1)\ \ -2}{3-0.1} = \dfrac{3-2}{3} = \dfrac{1}{3}$

(iv) $\lim\limits_{x\to 0} \dfrac{(a+x)^2\sin(a+x) - a^2\sin a}{x}\ \left(\dfrac{0}{0}\text{form}\right)$ using L' Hospital rule

$= \lim\limits_{x\to 0} \dfrac{2(a+x)\sin(a+x)\ \ +(a+x)^2\cos(a+x)}{1} = 2a\sin a + a^2\cos a$

(v) $\lim\limits_{x\to 0} \dfrac{e^{bx}-e^{ax}}{x}$, where $0 < a < b$

$= \lim\limits_{x\to 0} \dfrac{\left(1+\dfrac{bx}{1\ !}+\dfrac{(bx)^2}{2!}+....\right)-\left(1+\dfrac{ax}{1\ !}+\dfrac{(ax)^2}{2\ !}+....\right)}{x} = \lim\limits_{x\to 0} \dfrac{(b-a)x+\dfrac{1}{2}x^2(b^2-a^2)\ \ +\ \ ....}{x}$

$= b-a$

(vi) $\lim\limits_{x\to 0} \dfrac{x\ \left(e^{2+x}-e^2\right)}{1-\cos x} = \lim\limits_{x\to 0} \dfrac{x.e^2(e^x-1)}{2\ \ \sin^2\dfrac{x}{2}} = \lim\limits_{x\to 0} \dfrac{2e^2\left(\dfrac{e^x-1}{x}\right)}{\left(\dfrac{\sin\dfrac{x}{2}}{\dfrac{x}{2}}\right)^2} = 2e^2.\dfrac{1}{(1)^2} = 2e^2$

(vii) $\lim\limits_{x\to 0} \dfrac{\ell n(1+3x)}{3^x-1} = \lim\limits_{x\to 0} \dfrac{\left[\dfrac{3x}{1}-\dfrac{(3x)^2}{2}+\dfrac{(3x)^3}{3}-........\right]}{\left[1+\dfrac{x\ell n3}{1!}+\dfrac{(x\ell n3)^2}{2!}+......\right]-1} = \lim\limits_{x\to 0} \dfrac{3x\ \left[1-\dfrac{3x}{2}+......\right]}{x\ \ \ell n3\ \left[1+\dfrac{(x\ell n3)}{2!}+.....\right]} = \dfrac{3}{\ell n3}$

(viii) $\lim\limits_{x\to 0} \dfrac{\ell n\ \ (2+x)+\ell n\ \ \ 0.5}{x} = \lim\limits_{x\to 0} \dfrac{\ell n(2+x)\ \ (0.5)}{x} = \lim\limits_{x\to 0} \dfrac{\ell n\left(1+\dfrac{x}{2}\right)}{x} = \lim\limits_{x\to 0} \dfrac{\ell n\left(1+\dfrac{x}{2}\right)}{\dfrac{x}{2}}.\dfrac{1}{2} = \dfrac{1}{2}$

(ix) $\lim\limits_{x\to 2} \dfrac{x^n-2^n}{x-2} = 80,\ n\in N$

L.H.L. = R.H.L. = 80 so R.H.L. = 80

$\Rightarrow\quad 80 = \lim\limits_{h\to 0} \dfrac{(2+h)^n-2^n}{(2+h)\ \ -\ \ 2} \quad\Rightarrow\quad 80 = \lim\limits_{h\to 0} \dfrac{2^n\left[\left[1+\dfrac{h}{2}\right]^n-1\right]}{h}$

$\Rightarrow\quad 80 = \lim\limits_{h\to 0} \dfrac{2^n\left[1+\dfrac{nh}{2}+\dfrac{n(n-1)}{2!}.\dfrac{n^2}{4}+.....1\right]}{h} \quad\Rightarrow\quad n.\ 2^{n-1} = 80 \quad\Rightarrow\quad n = 5$

(x) $\lim\limits_{x\to 0} \dfrac{\sqrt{\dfrac{1-\cos 2x}{2}}}{x} = \lim\limits_{x\to 0} \dfrac{|\sin x|}{x} \Rightarrow$ L.H.L. = – 1 & R.H.L. = 1

(xi) $\lim\limits_{x \to 1} \dfrac{(\ell n(1+x) - \ell n2)\ \ (3.4^{x-1} - 3x)}{[(7+x)^{1/3} - (1+3x)^{1/2}]\ \ \sin(x-1)} = \lim\limits_{h \to 0} \dfrac{[\ell n\ \ (2+h) - \ell n2]\ \ \ [3.4^{h} - 3 - 3h]}{[(8+h)^{1/3} - (4+3h)^{1/2}]\ \ \sin\ \ h}$

$$= \lim_{h \to 0} \frac{\ell n\ \left(1+\frac{h}{2}\right)\ \ \left[3(4^h-1)\ \ \ -3h\right]}{2\left[\left(1+\frac{h}{8}\right)^{1/3} - \left(1+\frac{3h}{4}\right)^{1/2}\right]\sin\ \ h} = \lim_{h \to 0} \frac{\frac{\ell n\ \left(1+\frac{h}{2}\right)}{(h/2)} \cdot \left[\frac{3(4^h-1)}{h} - 3\right]}{4\ \ \frac{\sin\ \ h}{h}\left[\frac{\left(1+\frac{h}{24}\right) - \left(1+\frac{3h}{8}\right)}{h}\right]}$$

$$= \frac{1\ \ .(3\ell n\ \ 4-3)}{4\ \left(-\frac{1}{3}\right)} = -\frac{9}{4}\ \ (\ell n\ 4-1) = -\frac{9}{4}\ \ \ \ell n\ \ \left(\frac{4}{e}\right)$$

B-3. **(i)** $\lim\limits_{x\to\infty}\left(\dfrac{1}{x^2}+\dfrac{2}{x^2}+....+\dfrac{x}{x^2}\right) = \lim\limits_{x\to\infty}\dfrac{(1+2+.....+x)}{x^2} = \lim\limits_{x\to\infty}\dfrac{x(x+1)}{2x^2} = \lim\limits_{x\to\infty}\dfrac{1}{2}\left[1+\dfrac{1}{x}\right] = \dfrac{1}{2}$

(ii) $\lim\limits_{n \to \infty} \dfrac{\sqrt{n^3-2n^2+1}+\sqrt[3]{n^4+1}}{\sqrt[4]{n^6+6n^5+2}-\sqrt[5]{n^7+3n^3+1}} = \lim\limits_{n\to\infty} \dfrac{n^{\frac{3}{2}}\ \sqrt{1-\frac{2}{n}\ \ +\frac{1}{n^3}}+\ \ n^{\frac{4}{3}}\sqrt{1+\frac{1}{n^4}}}{n^{\frac{3}{2}}\sqrt{1+\frac{6}{n}+\frac{2}{n^6}}-n^{\frac{7}{5}}\sqrt{1+\frac{3}{n^4}+\frac{1}{n^7}}}$

$$= \lim_{n\to\infty} \frac{\sqrt{1-\frac{2}{n}\ \ +\frac{1}{n^3}}+\ \ n^{-\frac{1}{6}}\sqrt{1+\frac{1}{n^4}}}{\sqrt{1+\frac{6}{n}+\frac{2}{n^6}}-n^{-\frac{1}{10}}\sqrt{1+\frac{3}{n^4}+\frac{1}{n^7}}} = \frac{1+0}{1-0} = 1$$

(iii) $\lim\limits_{x\to\infty}\left(\sqrt{x^2-8x}+x\right) = (\infty + \infty) = \infty$

(iv) $\lim\limits_{x \to -\infty} \dfrac{x^5\ \tan\ \left(\frac{1}{\pi x^2}\right)\ +\ 3\ \ |\,x\,|^2\ +\ \ 7}{|\,x\,|^3\ +7\,|\,x\,|\ +\ \ 8} = \lim\limits_{x \to \infty} \dfrac{-x^5\tan\left(\frac{1}{\pi x^2}\right)+3x^2+7}{x^3+7x+8}$

$$= \lim_{x \to \infty} = \frac{\frac{3}{x}+\frac{7}{x^3}-\frac{\tan\left(\frac{1}{\pi x^2}\right)}{\frac{1}{\pi x^2}}\times\frac{1}{\pi}}{1+\frac{7}{x^2}+\frac{8}{x^3}} - \frac{1}{\pi}$$

B-4. **(i)** $\lim\limits_{x \to \infty}\left((x+1)^{\frac{2}{3}}-(x-1)^{\frac{2}{3}}\right) = \lim\limits_{x \to \infty}\dfrac{\left[(x+1)^{\frac{2}{3}}-(x-1)^{\frac{2}{3}}\right]\left[(x+1)^{\frac{4}{3}}+(x+1)^{\frac{2}{3}}(x-1)^{\frac{2}{3}}+(x-1)^{\frac{4}{3}}\right]}{\left[(x+1)^{\frac{4}{3}}+(x+1)^{\frac{2}{3}}(x-1)^{\frac{2}{3}}+(x-1)^{\frac{4}{3}}\right]}$

$$= \lim_{x \to \infty}\frac{(x+1)^2-(x-1)^2}{\left[(x+1)^{\frac{4}{3}}+(x+1)^{\frac{2}{3}}(x-1)^{\frac{2}{3}}+(x-1)^{\frac{4}{3}}\right]} = \lim_{x \to \infty}\frac{4x}{(x+1)^{\frac{4}{3}}\left[1+\left(\frac{x-1}{x+1}\right)^{\frac{2}{3}}+\left(\frac{x-1}{x+1}\right)^{\frac{4}{3}}\right]}$$

$$= \lim_{x \to \infty}\frac{4x}{(x+1)(x+1)^{\frac{1}{3}}\left[1+\left(\frac{x-1}{x+1}\right)^{\frac{2}{3}}+\left(\frac{x-1}{x+1}\right)^{\frac{4}{3}}\right]} = \frac{4}{(1+0)\times(\infty)\ \ \times[1+1+1]} = 0$$

(ii) $\lim\limits_{x\to a} \dfrac{(x+2)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{x-a}$ R.H.L. = $\lim\limits_{x\to a^+} \dfrac{(x+2)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{x-a} = \lim\limits_{h\to 0} \dfrac{(a+2+h)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{a+h-a}$

$$= \lim\limits_{h\to 0} \frac{(a+2)^{\frac{5}{2}}\left[\left(1+\dfrac{h}{a+2}\right)^{\frac{5}{2}}-1\right]}{h} = \lim\limits_{h\to 0} \frac{(a+2)^{\frac{5}{2}}\left[1+\dfrac{5}{2}.\ \dfrac{h}{a+2}+\dfrac{\left(\frac{5}{2}\right)\ \left(\frac{3}{2}\right)}{2!}\ .\dfrac{h^2}{(a+2)^2}+.....-1\right]}{h}$$

$$= \frac{5}{2}\ (a+2)^{\frac{3}{2}}$$

$$\text{L.H.L.} = \lim\limits_{h\to 0}\frac{(a+2-h)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{-h} = \lim\limits_{h\to 0} \frac{(a+2)^{\frac{5}{2}}\left[1-\dfrac{5}{2}\left(\dfrac{h}{a+2}\right)+\dfrac{\left(\frac{5}{2}\right)\ \left(\frac{3}{2}\right)}{2!}\left(\dfrac{h}{a+2}\right)^2+....-1\right]}{-h}$$

$$= \frac{5}{2}\ (a+2)^{\frac{3}{2}}$$

L.H. L. = R.H.L. So $\lim\limits_{x\to a} \dfrac{(x+2)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{(x-a)} = \dfrac{5}{2}\ (a+2)^{\frac{3}{2}}$

(iii) $\lim\limits_{x\to\infty} \left\{\cos\left(\sqrt{x+1}\right) - \cos\left(\sqrt{x}\right)\right\} = \lim\limits_{x\to\infty} 2\sin\left(\dfrac{\sqrt{x}+\sqrt{x+1}}{2}\right).\sin\left(\dfrac{\sqrt{x}-\sqrt{x+1}}{2}\right)$

$$= \lim\limits_{x\to\infty} 2\ \sin\left(\frac{\sqrt{x}+\sqrt{x+1}}{2}\right)\ .\ \sin\left(\frac{x-x-1}{2\ \ .\ \ (\sqrt{x}+\sqrt{x+1})}\right)$$

$$=2\lim\limits_{x\to\infty} \sin\left(\frac{\sqrt{x}+\sqrt{x+1}}{2}\right)\ .\ \lim\limits_{x\to\infty} \sin\left[\frac{-1}{2\ \ .\ \ (\sqrt{x}+\sqrt{x+1})}\right]$$

= 2x (oscillating –1 to 1) × 0 2x = 0

(iv) $\lim\limits_{x\to\infty} \left(((x+1)\ (x+2)\ (x+3)\ (x+4))^{\frac{1}{4}}-x\right) = \lim\limits_{x\to\infty} x\left(\left(1+\dfrac{1}{x}\right)\left(1+\dfrac{2}{x}\right)\left(1+\dfrac{3}{x}\right)\left(1+\dfrac{4}{x}\right)\right)^{\frac{1}{4}}-x$

$$= \lim\limits_{x\to\infty} x\left[1+\frac{10}{x}+\frac{\Sigma\ \ 1.2}{x^2}+\frac{\Sigma\ \ 1.2.3}{x^3}+\frac{24}{x^4}\right]^{\frac{1}{4}}-x$$

$$= \lim\limits_{x\to\infty} x\left[1+\frac{\left(\dfrac{10}{x}+\dfrac{\Sigma\ \ 1.2}{x^2}+\dfrac{\Sigma\ \ 1.2.3}{x^3}+\dfrac{24}{x^4}\right)}{4}+....\right]-x = \frac{10}{4}=\frac{5}{2}$$

B-5. **(i)** $\lim\limits_{x\to 2} \dfrac{(x+2)^{\frac{1}{2}} - (15x+2)^{\frac{1}{5}}}{(7x+2)^{\frac{1}{4}} - x}$

Let x = 2 + h = $\lim\limits_{h\to 0} \dfrac{(4+h)^{\frac{1}{2}} - (32+15h)^{\frac{1}{5}}}{(16+7h)^{\frac{1}{4}} - (2+h)}$ = $\lim\limits_{h\to 0} \dfrac{2\left[1+\frac{h}{4}\right]^{\frac{1}{2}} - 2\left[1+\frac{15h}{32}\right]^{\frac{1}{5}}}{2\left[1+\frac{7h}{16}\right]^{\frac{1}{4}} - (2+h)}$

$$= \lim_{h\to 0} \frac{2\left[1+\frac{h}{8}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}\left(\frac{h^2}{16}\right)+....\right] - 2\left[1+\frac{3h}{32}+\frac{\left(\frac{1}{5}\right)\left(-\frac{4}{5}\right)}{2}\left(\frac{15h}{32}\right)^2+....\right]}{2\left[1+\frac{7h}{64}+......\right]-(2+h)}$$

$$= \lim_{h\to 0} = \frac{h\left(\frac{1}{4}-\frac{3}{16}\right)+h^2\left(-\frac{1}{64}-\frac{9}{256}\right)+...}{h\left(\frac{7}{32}-1\right)+.....} = \frac{\frac{1}{4}-\frac{3}{16}}{\frac{7}{32}-1} = -\frac{2}{25}$$

(ii) $\lim\limits_{x\to 0} \dfrac{e^x - 1 - \sin x - \dfrac{\tan^2 x}{2}}{x^3}$ $\lim\limits_{h\to 0} \dfrac{\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+...\right)-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+...\right)-\frac{1}{2}\left(x+\frac{x^3}{3}+...\right)^2}{x^3}$

$$= \lim_{h\to 0} \frac{x^2\left(\frac{1}{2}-\frac{1}{2}\right) + x^3\left(\frac{1}{6}-\frac{1}{6}\right)+x^4\left(\frac{1}{4\,!}-\frac{1}{3}\right)+....}{x^3} = \frac{1}{6}+\frac{1}{6}=\frac{1}{3}$$

B-6 $\lim\limits_{x\to 0} \dfrac{a+b\sin x - \cos x + ce^x}{x^3}$

$$\lim_{x\to 0} \frac{a+b\left[x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-....\right] - \left[1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-....\right] + c\left[1+\frac{x}{1\,!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+....\right]}{x^3}$$

$$\lim_{x\to 0} \frac{(a-1+c)+x(b+c)+x^2\left(\frac{1}{2}+\frac{c}{2}\right)+x^3\left(\frac{-b+c}{6}\right)+x^4\left(\frac{-1}{4!}+\frac{c}{4!}\right)+.....}{x^3}$$

limit exists so

a + c – 1 = 0

b + c = 0

$\frac{1}{2}+\frac{c}{2} = 0 \quad \Rightarrow \quad c = -1$

so b = 1

a = 2

value is = $\dfrac{-b+c}{6} = \dfrac{-1-1}{6} = -\dfrac{1}{3}$

B-7. **(i)** $\lim_{x\to 0}\dfrac{1+ax\sin x - b\cos x}{x^4}$

$= \lim_{x\to 0}\dfrac{1 + ax\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-........\right)-b\left(1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-........\right)}{x^4}$

$= \lim_{x\to 0}\dfrac{(1-b)+x^2\left(a+\dfrac{b}{2}\right)+x^4\left(-\dfrac{a}{6}-\dfrac{b}{24}\right)+x^6\left(\dfrac{a}{120}+\dfrac{b}{720}\right)+...........}{x^4}$

Limit exist and have a finite value So, $1 - b = 0$ & $a + \dfrac{b}{2} = 0$, $b = 1$, $a = -\dfrac{1}{2}$

(ii) $\lim_{x\to\infty}\left(\sqrt{x^4+ax^3+3x^2+bx+2}-\sqrt{x^4+2x^3-cx^2+3x-d}\right) = 4$

$\Rightarrow \lim_{y\to 0}\left\{\dfrac{\sqrt{1+ay+3y^2+by^3+2y^4}-\sqrt{1+2y-cy^2+3y^3-dy^4}}{y^2}\right\} = 4$

$\Rightarrow \lim_{y\to 0}\dfrac{(a-2)y+(3+c)y^2+(b-3)y^3+(2+d)y^4}{2y^2} = 4 \Rightarrow a-2=0$ & $\dfrac{3+c}{2}=4 \Rightarrow c = 5$

$\Rightarrow$ $a = 2$, $c = 5$, $b \in R$, $d \in R$

(iii) $\lim_{x\to 0}\dfrac{ax\left(1+x+\dfrac{x^2}{2!}+.........\right)-b\left(x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-........\right)+cx\left(1-x+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+.......\right)}{x^2\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}..........\right)} = 2$

$\Rightarrow \lim_{x\to 0}\dfrac{(a-b+c)+x\left(a+\dfrac{b}{2}-c\right)+x^2\left(\dfrac{a}{2}-\dfrac{b}{3}+\dfrac{c}{2}\right)+x^3 \text{ and term containing higher power}}{x^2-\dfrac{x^4}{3!}+\dfrac{x^6}{5!}-..........} = 2$

then $a - b + c = 0$ .....(1)

$2a + b - 2c = 0$ .....(2)

$3a - 2b + 3c = 12$ .....(3)

On solving (1), (2), (3)

we get $a = 3$, $b = 12$, $c = 9$

B-8. $\lim_{x\to 0}\left(\dfrac{\ell n\,(1+x)^{1+x}}{x^2}-\dfrac{1}{x}\right) = \lim_{x\to 0}\left[\dfrac{(1+x)\,\ell n\,(1+x)-x}{x^2}\right] = \lim_{x\to 0}\left[\dfrac{(1+x)\left(x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-....\right)-x}{x^2}\right]$

$= \lim_{x\to 0}\left(\dfrac{\dfrac{x^2}{2}-\dfrac{1}{6}x^3+......}{x^2}\right) = \dfrac{1}{2}$

B-9. $\lim_{x\to 4}\dfrac{(\cos\alpha)^x-(\sin\alpha)^x-\cos 2\alpha}{x-4}$, $\alpha \in \left(0, \dfrac{\pi}{2}\right)$

$= \lim_{x\to 4}\dfrac{(\cos\alpha)^x-(\sin\alpha)^x-(\cos^2\alpha-\sin^2\alpha)\,(\cos^2\alpha+\sin^2\alpha)}{x-4}$

$= \lim_{x\to 4}\dfrac{(\cos\alpha)^x-(\sin\alpha)^x-(\cos^4\alpha-\sin^4\alpha)}{x-4}$

$= \lim_{x\to 4}\dfrac{(\cos\alpha)^4\,[(\cos\alpha)^{x-4}-1]-(\sin\alpha)^4\,[(\sin\alpha)^{x-4}-1]}{x-4} = \cos^4\alpha\,\ell n(\cos\alpha) - \sin^4\alpha\,\ell n\,(\sin\alpha)$= R.H.S.

B-10. Required = $\left.\dfrac{d^2(\tan x)}{dx}\right|_{x=a} = 2\sec^2 x \tan x$ at x = a

$\Rightarrow \quad \lim\limits_{h\to 0}\dfrac{\tan(a+2h)-2\tan(a+h)+\tan a}{h^2}$ = 2(sec^{2}a) tan

Section (C)

C-1 **(i)** $\ell im_{x\to 0^+} (x)^{x^2}$ $\quad ((0)^0$ form)

Let y = $\ell im_{x\to 0^+} (x)^{x^2}$

$\Rightarrow \quad y = e^{\ell im_{x\to 0^+} x^2 \ell nx} \quad \Rightarrow \quad y = e^{\ell im_{x\to 0^+}\frac{\ell nx}{\frac{1}{x^2}}} \quad \Rightarrow \quad y = e^{\ell im_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{2}{x^3}}} = e^0 = 1$

(ii) $\ell im_{x\to \frac{\pi}{2}^-}$ (tan x)$^{\cos x}$ $\quad (\infty^0$ form) $\Rightarrow y = e^{\ell im_{x\to \frac{\pi}{2}^-}\cos x.(\ell n\tan x)} \quad \Rightarrow y = e^{\ell im_{x\to \frac{\pi}{2}^-}\frac{\ell n\tan x}{\sec x}}$

$\Rightarrow y = e^{\ell im_{x\to \frac{\pi}{2}^-}\frac{1}{\tan x}\times\frac{\sec^2 x}{\sec x\ \tan x}} \quad \Rightarrow y = e^{\ell im_{x\to \frac{\pi}{2}^-}\frac{\cos x}{\sin^2 x}} \quad \Rightarrow y = e^0 = 1$

(iii) $\ell im_{x\to 1^-}$ $([x])^{1-x}$; $\quad \ell im_{h\to 0}$ $[1-h]^{1-(1-h)}$; $\quad \ell im_{h\to 0}$ $(0)^h = 0$

(iv) Let y = $x-\dfrac{\pi}{2}$; $\ell im_{y\to 0^+} e^{-\cot y}$ = $\ell im_{y\to 0^+}\dfrac{1}{e^{\cot y}}$ = 0

C-2. **(i)** $\ell im_{x\to \frac{\pi}{4}}$ (tan x)$^{\tan 2x}$ $\quad (1^\infty$ form)

$= e^{\ell im_{x\to\frac{\pi}{4}}(\tan x-1)\tan 2x} \quad = e^{\ell im_{x\to\frac{\pi}{4}}\frac{-2\tan x}{1+\tan x}} \quad = e^{-1} = \dfrac{1}{e}$

(ii) $\ell im_{x\to\infty}\left(\dfrac{1+2x}{1+3x}\right)^x = \ell im_{x\to\infty}\left(\dfrac{\frac{1}{x}+2}{\frac{1}{x}+3}\right)^x = \left(\dfrac{2}{3}\right)^\infty = 0$

(iii) $\ell im_{x\to 1}(1+\ell nx)^{\sec\frac{\pi x}{2}}$ $\quad (1^\infty$form) = $e^{\ell im_{x\to 1}(\ell nx).\sec\frac{\pi x}{2}}$

$= e^{\ell im_{x\to 1}\frac{\ell nx}{\cos\frac{\pi x}{2}}}$ $\left(\dfrac{0}{0}\right.$ form) = $e^{\lim_{x\to 1}\frac{\frac{1}{x}}{-\frac{\pi}{2}\sin\frac{\pi x}{2}}} \quad = e^{\left(\frac{-2}{\pi}\right)}$

(iv) $\ell im_{x\to 0}\left(\dfrac{1+\tan x}{1-\tan x}\right)^{\frac{1}{x}} = \ell im_{x\to 0}\left(1+\dfrac{2\tan x}{1-\tan x}\right)^{\frac{1}{x}} = \ell im_{x\to 0}\exp\left[\dfrac{2\tan x}{x(1-\tan x)}\right] = e^2$

C-3 $\ell im_{x\to 1}\left(1+ax+bx^2\right)^{\frac{c}{x-1}} = e^3$

it takes $(1 + a + b)^\infty$ form so 1 + a + b = 1 $\Rightarrow$ $\quad$ a + b = 0 $\quad$ ....(i)

$\ell im_{x\to 1}(1+ax+bx^2)^{\frac{c}{(x-1)}} = e^3 \quad \Rightarrow e^{\ell im_{x\to 1}(1+ax+bx^2-1).\frac{c}{(x-1)}} = e^3$

$\Rightarrow e^{\ell im_{x\to 1}\frac{(ax+bx^2)c}{(x-1)}} = e^3 \quad \Rightarrow \quad e^{\ell im_{x\to 1}(a+2bx)c} = e^3$

$\Rightarrow \quad e^{(a+2b)c} = e^3 \quad \Rightarrow \quad (a + 2b)c = 3 \quad \Rightarrow \quad bc = 3$

C-4. **(i)** $\lim\limits_{x\to\infty} \dfrac{x \quad \ell n \quad \left(1+\dfrac{\ell n \quad x}{x}\right)}{\ell n \quad x} = \lim\limits_{x\to\infty} \dfrac{\ell n\left(1+\dfrac{\ell nx}{x}\right)}{\left(\dfrac{\ell nx}{x}\right)}$

$\because \quad \lim\limits_{x\to\infty} \dfrac{\ell nx}{x} = 0$ and $\lim\limits_{f(x)\to 0} \dfrac{\ell n(1+f(x))}{f(x)} = 1 = 1$

(ii) $\lim\limits_{x\to\infty} \dfrac{e^x \sin\left(\dfrac{x^n}{e^x}\right)}{x^n} = \lim\limits_{x\to\infty} \dfrac{\sin\left(\dfrac{x^n}{e^x}\right)}{\left(\dfrac{x^n}{e^x}\right)}$

$\because \quad \lim\limits_{x\to\infty} \dfrac{x^n}{e^x} = 0$ and $\lim\limits_{f(x)\to 0} \dfrac{\sin \quad (f(x))}{f(x)} = 1 = 1$

C-5. $\lim\limits_{n\to\infty} \dfrac{[1 \ . \ 2x]+[2 \ . \ 3x]+.....+[n \ . \ (n+1) \ x]}{n^3}$

$(1.2)x - 1 < [1.2x] \le (1.\ 2)x$

$(2.3)x - 1 < [2.3x] \le (2.3)x$

$n(n + 1)\ x - 1 < [n\ (n + 1)x] \le n(n + 1)x$

so $\quad (1.2)x + (2.3)x + ...\ n\ (n + 1)x - n$

$< [1.2\ x] + [2.\ 3\ x] + ...\ [n(n +1)x]$

$\le (1.\ 2)\ x + (2.\ 3)\ x + ...\ n(n +1)x$

$x\ .\ (\Sigma n^2 + \Sigma n) - n \le [1.\ 2x] + [2.\ 3x] +......[n(n+1)x] \le x\ (\Sigma n^2 + \Sigma n)$

$\Rightarrow \quad \lim\limits_{n\to\infty} \dfrac{x.\left[\dfrac{n(n+1)\quad(2n+1)}{6}+\dfrac{n\quad(n\quad+1)}{2}\right]-n}{n^3} < \lim\limits_{n\to\infty} \dfrac{[1.2 \quad x] \quad + \quad [2.3 \quad x]+.....+[n(n+1)x]}{n^3}$

$\le \lim\limits_{n\to\infty} \dfrac{x\left[\dfrac{n(n+1)\quad(2n+1)}{6}+\dfrac{n\quad(n\quad+1)}{2}\right]}{n^3}$

$\lim\limits_{n\to\infty} \quad \times \quad \left[\dfrac{1.\left(1+\dfrac{1}{n}\right)\quad\left(2+\dfrac{1}{n}\right)}{6}+\dfrac{\left(\dfrac{1}{n}+\dfrac{1}{n^2}\right)}{2}\right]-\dfrac{1}{n^2} \quad < \lim\limits_{n\to\infty} \dfrac{[1.2 \quad x] \quad + \quad [2.3 \quad x]+.....+[n(n+1)x]}{n^3}$

$\le \lim\limits_{n\to\infty} \times \left[\dfrac{1.\left(1+\dfrac{1}{n}\right)\quad\left(2+\dfrac{1}{n}\right)}{6}+\dfrac{\left(\dfrac{1}{n}+\dfrac{1}{n^2}\right)}{2}\right] \dfrac{x}{3} < \lim\limits_{n\to\infty} \dfrac{[1.2 \quad x] \quad + \quad [2.3 \quad x]+.....+[n(n+1)x]}{n^3} \le \dfrac{x}{3}$

so $\lim\limits_{n\to\infty} \dfrac{[1.2 \quad x] \quad + \quad [2.3 \quad x]+.....+[n(n+1)x]}{n^3} = \dfrac{x}{3}$

C-6. $f(x) = \lim\limits_{n \to \infty} \dfrac{x^{2n}-1}{x^{2n}+1}$

case (i) when x = 1

$f(x) = \lim\limits_{n \to \infty} \dfrac{1-1}{1+1} = 0$

case (ii) when x > 1

$f(x) = \lim\limits_{n \to \infty} \dfrac{1-\left(\frac{1}{x}\right)^{2n}}{1+\left(\frac{1}{x}\right)^{2n}} = \dfrac{1-0}{1+0} = 1$

case (iii) when x < 1

$f(x) = \lim\limits_{n \to \infty} \dfrac{x^{2n}-1}{x^{2n}+1} = \dfrac{0-1}{0+1} = -1$

range of f(x) is {–1, 0, 1}

Section (D)

D-1. f (x) is continous at x = 0 therefore

$\lim\limits_{x\to0^-} f(x) = f(0) = \lim\limits_{x\to0^+} f(x)$; $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} \left\{\dfrac{(a+1)\ \sin\ (a+1)x}{(a+1)x} + \dfrac{\sin x}{x}\right\} = a + 2$

and $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} \dfrac{(1+bx)^{\frac{1}{2}}-1}{bx} = \dfrac{1}{2}$ here b should not be equal to zero. so a + 2 = c = 1/2 and b $\neq$ 0

D-2. $\lim\limits_{x\to\frac{\pi}{2}^-} f(x) = \lim\limits_{x\to\frac{\pi}{2}^-} \dfrac{1-\sin^3 x}{3\cos^2 x} = \lim\limits_{h\to0} \dfrac{1-\cos^3 h}{3\sin^2 h}\left[\text{put } x = \dfrac{\pi}{2} - h\right] = \lim\limits_{h\to0} \dfrac{1-\left(1-\frac{h^2}{2\ !}+\frac{h^4}{4\ !}-....\right)^3}{3h^2} = \dfrac{1}{2}$

and $\lim\limits_{x\to\frac{\pi}{2}^+} f(x) = \lim\limits_{x\to\frac{\pi}{2}^+} \dfrac{b(1-\sin x)}{(\pi-2x)^2} = \lim\limits_{h\to0} \dfrac{b(1-\cos h)}{4h^2} = \dfrac{b}{8}$ so $\dfrac{1}{2} = \dfrac{b}{8} = a$

D-3. **(i)** h(x) = {x} [x]

at x = 1   h(1) = 0   LHL = 0 RHL = 0

at x = 2   h(2) = 0   LHL = 1 RHL = 0

(ii) h(x) = {x} + [x] = x at x = 1   continuous

(iii) h(x) = {x} – [x] = x – 2[x] discontinuous at x = 1

(iv) $h(x) = \sqrt{\{x\}} + [x]$

at x = 1   h(1) = 1   LHL = 1 RHL = 1

at x = 2   h(2) = 2   LHL = 2 RHL = 2

D-4. f (x) = (x + 2) (x – 2) (x – 3)

$h(x) = \begin{cases} (x+2)\ (x-2), & x \neq 3 \\ k, & x = 3 \end{cases}$

for continuity $k = \lim\limits_{x\to3} h(x) = 5$

h(x) = (x + 2) (x – 2) = $x^2 - 4$ which is even $\forall$ x $\in$ R

D-5. $\lim\limits_{x\to 0}\dfrac{\sin 3x + A\sin 2x + B\sin x}{x^5} = 1$

$\lim\limits_{x\to 0}\ \dfrac{1}{x^5}\left[\left(3x-\dfrac{(3x)^3}{3\ !}+\dfrac{(3x)^5}{5\ !}-\dfrac{(3x)^7}{7\ !}+...\right)+A\left(2x-\dfrac{(2x)^3}{3\ !}+\dfrac{(2x)^5}{5\ !}-\dfrac{(2x)^7}{7\ !}+...\right)+B\left(x-\dfrac{x^3}{3\ !}+\dfrac{x^5}{5\ !}-\dfrac{x^7}{7\ !}+...\right)\right]$

$\lim\limits_{x\to 0}\ \dfrac{1}{x^4}(3+2A+B)-\dfrac{1}{x^2 3\ !}(27+8A+B)+\dfrac{1}{5\ !}(243+32A+B)$ – term containing x this Limit exist ans finite if 3 +2A + B = 0 and 27 +8A + B = 0

In this case $\lim\limits_{x\to 0} f(x) = \dfrac{1}{5\ !}(243+32A+B)$ $\Rightarrow$ A = – 4, B = 5 and for continuity f(0) = $\lim\limits_{x\to 0} f(x) = 1$

D6. f(x) g(x) = $\begin{cases}1 & x \neq 2\\ 2 & x = 2\end{cases}$

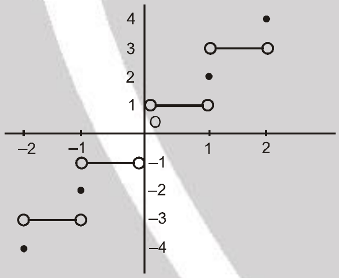
Section (E)

E-1. **(i)** f(x) is not defined at x = 2, 3 **(ii)** f(x) is not defined when |x| = 1

(iii) f(x) is always defined **(iv)** $\dfrac{\pi x}{2} \neq (2n + 1)\ \dfrac{\pi}{2}$

E-2. f(x) = x + {–x} + [x] Since {–x} = $\begin{cases}1-\{x\}, & x \notin I\\ 0\ , & x \in I\end{cases}$

= $\begin{cases}x+1-\{x\}+[x] & , & x\notin I\\ x+[x] & , & x \in I\end{cases}$ = $\begin{cases}1+2\ \ [x] & , & x\notin I\\ 2x & , & x \in I\end{cases}$



Curve of y = f(x) discontinuous at all integers in [–2, 2]

E-3. fog (x) = $\dfrac{\tan^2 x+1}{\tan^2 x-1}$ discontinous when

(i) x = (2n + 1) $\dfrac{\pi}{2}$ $x \in I$

(ii) tan x = ± 1

x = nπ ± $\dfrac{\pi}{4}$

E-4. f(x) = $\begin{cases}1\ +\ x & ,\ 0 \le\ x \le 2\\ 3\ -\ x & ,\ 2 < x \le 3\end{cases}$

y(x) = fof(x) = $\begin{cases}1\ + f(x) & \text{when} & 0 \le\ f(x)\ \le\ 2\\ 3\ - f(x) & \text{when} & 2 <\ f(x)\ \le\ 3\end{cases}$ = $\begin{cases}1+(1+x) & \text{when} & 0\le x\le 1\\ 1+(3-x) & \text{when} & 2\le x\le 3\\ 3-(1+x) & \text{when} & 1\le x\le 2\end{cases}$

$g(1^-) = g(1) = 3 \neq g(1^+) = 1$

$g(2^-) = g(2) = 0 \neq g(2^+) = 2$

E-5. $u = \frac{1}{x+2}$ is discontinuous at x = – 2

$f(u) = \frac{3}{2u^2+5u-3} = \frac{3}{2u^2+6u-u-3} = \frac{3}{(2u-1)\ (u+3)}$ is

discontinuous at $u = \frac{1}{2}$ & –3

$\therefore \quad \frac{1}{x+2} = \frac{1}{2}$ and $\frac{1}{x+2} = -3 \Rightarrow x = 0$ and $x = -\frac{7}{3}$

Hence y = f(u) is discontinous at $x = -\frac{7}{3}, -2, 0$

E-6. f(x) is continuous and $\frac{7}{3} \in [f(-2), f(2)]$, by intermediate value theorem (IVT), there exists a point $c \in (-2, 2)$ such that $f(c) = \frac{7}{3}$

E-7. $g(x) = (|x-1| + |4x-11|)\ [(x-1)^2 - 3] = (|x-1| + |4x-11|)\ ([(x-1)^2] - 3)$

Now $|x-1| + |4x-11|$ is continuous every where & $[(x-1)^2 - 3]$ is discontinuous at x = 1, 2; $\sqrt{2}+1$

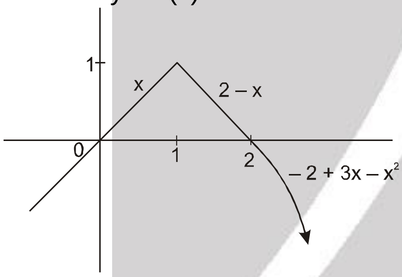
At x = 1, g(x) is continuous

At x = 2, g(x) is discontinuous

At $x = (\sqrt{2} + 1)$, g(x) is discontinuous

Section (F)

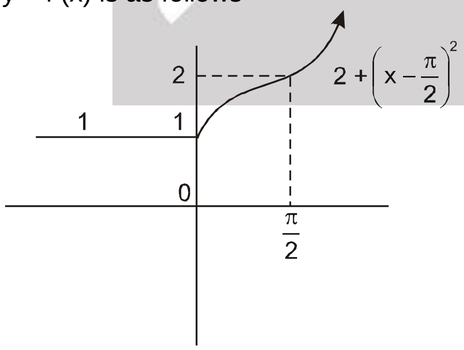
F-1. Curve of y = f (x) is as follows



It is continous every where not differentiable at x = 1

LHD (x = 2) = – 1 = RHD (x = 2) diff. at x = 2

F-2. y = f (x) is as follows



continous every where

non-differentiable at x = 0

F-3. $f(x) = |x| \cos x = \begin{cases} x\ \cos\ x\ , & x \geq 0 \\ -x\cos x\ , & x < 0 \end{cases}$

f (x) is non differentiable at x = 0

F-4. $f(x) = \begin{cases} x^m \sin\left(\frac{1}{x}\right) & ; \quad x > 0 \\ 0 & ; \quad x = 0 \end{cases}$

for continitity f(0) = 0 = RHL (x = 0)

$\lim\limits_{h\to 0} \; h^m \sin\left(\frac{1}{n}\right) = 0$

$\lim\limits_{h\to 0} \; h^m$ [a finite quantity between [–1, 1]] = 0

It hold only when $m > 0$

if $m \le 0$ neither continuous nor derivable for derivability $\lim\limits_{h\to 0} \frac{f(h)-f(0)}{h}$ = finite

$\lim\limits_{h\to 0} \; h^{(m-1)} \sin\left(\frac{1}{h}\right)$ it is finite and unique and equal to zero if m > 1 when m >1 continuous and derivable

if $0 < m \le 1$ continuous but not derivable

F-5. $f(x) = \sqrt{1-e^{-x^2}}$ $f(0) = 0$

LHD (x = 0) = $\lim\limits_{h\to 0} \frac{\sqrt{1-e^{-h^2}}-0}{-h} = \lim\limits_{h\to 0} -\sqrt{1+\frac{h^2}{2\,!}+\ldots} = -1$

RHD (x = 0) = $\lim\limits_{h\to 0} \frac{\sqrt{1-e^{-h^2}}-0}{h} = \lim\limits_{h\to 0} \sqrt{1+\frac{h^2}{2\,!}+\ldots} = 1$

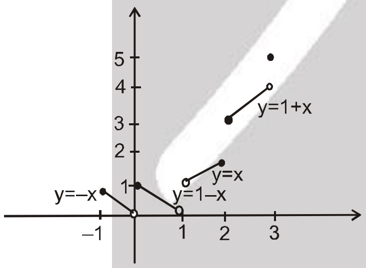
Not differentiable at x = 0

F-6. f(x) is continous at x =1 $\Rightarrow$ a–b = –1

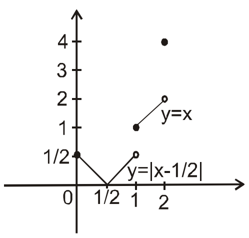
L.H.D. = $\lim\limits_{h \to 0} \frac{a(1-h)^2-b+1}{-h} = 2a$ R.H.D. = $\lim\limits_{h \to 0} \frac{\frac{-1}{1+h}+1}{h} = 1$ it gives $\Rightarrow a = \frac{1}{2}$, $b = \frac{3}{2}$

Section (G)

G-1. y = [x] + |1–x|



G-2. y = f(x)



G-3. [x] is dicontinuous at –1, 0
{x} is dicontinuous at 1
$\frac{1}{\log_4(x-3)}$ is dicontinuous at 3, 4

G-4. $|x-1|^{5/3}$ is differentiable for all x

Let $g(x) = \text{sgn}(x^{2/3}) + \left[\cos\left(\frac{x^2}{1+x^2}\right)\right] \Rightarrow g'(0) = \lim_{h\to 0} \frac{\text{sgn}(h^{2/3}) + \left[\cos\left(\frac{h^2}{1+h^2}\right)\right] - 1}{h} = \lim_{h\to 0} \frac{1+0-1}{h} = 0$

so differentiable at 0.

G-5. $h(x) = f(x)\,g(x) = \begin{cases} \frac{(e^x-e)(3-2x)}{2} & , x\in(0,1) \\ \frac{(e^x-e)}{2} & , x\in[1,2) \\ \frac{(e^x-e)(2x-3)}{2} & , x\in[2,3) \end{cases}$ $h'(x) = \begin{cases} \frac{e^x(3-2x)-2(e^x-e)}{2} & , x\in(0,1) \\ e/2 & , x=1 \\ \frac{e^x}{2} & , x\in(1,2) \\ \text{Not Defined} & , x=2 \\ \frac{e^x(2x-3)+2(e^x-e)}{2} & , x\in(2,3) \end{cases}$

Section (H)

H-1. $f(x + y) = f(x) + f(y)$ function should be $y = mx$

$f(1) = 2$

$\therefore \quad m = 2 \Rightarrow f(x) = 7x$

$$\sum_{r=1}^{n} f(r) = 2\sum_{r=1}^{n} r = \frac{2n(n+1)}{2}.$$

H-2. Given $f'(2) = 4$ $\lim_{x\to 0}\frac{f(1+\cos x)-f(2)}{\tan^2 x}\left[\frac{0}{0} \text{ form}\right]$

apply L′Hospital rule $\lim_{x\to 0}\frac{-\sin x \; f'(1+\cos x)}{2\tan x\sec^2 x} = \lim_{x\to 0}\frac{-\left(\frac{\sin x}{x}\right)f'(1+\cos x)}{2\left(\frac{\tan x}{x}\right)\sec^2 x} = -2$

H-3. $f : R \to R$ and $f(x + y) = f(x)\, f(y) \quad \forall\ x, y \in R$

Put $x = y = 0 \qquad f(0) = f^2(0)$ since $f(0) \neq 0$

$f(0) = 1$

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} f(x)\frac{[f(h)-1]}{h} = f(x)\, f'(0)$

Let $f(x) = y \Rightarrow \frac{dy}{dx} = y.f'(0)$

On solving $\ell ny = x\, f'(0) + c$

$y = f(x) = e^c.\, e^{x f'(0)}$

$\because f(0) = 1 \Rightarrow c = 0$

Thus $f(x) = e^{x.f'(0)}\ \forall\ x \in R$

H-4. $f(x)\ .\ f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \qquad \Rightarrow \qquad f(x) = 1 \pm x^n$

$f(3) = -26 \qquad \Rightarrow \qquad f(x) = 1 - x^3$

$f'(x) = -3x^2 \qquad$ or $\qquad f'(1) = -3$

H-5. Given $f(x + y) + f(x - y) = 2f(x).f(y)$ ....(1)

Putting y = x in equation (1), we get

$f(2x) + f(0) = 2(f(x))^2$ ....(2)

Putting y = – x in equation (1), we get

$f(0) + f(2x) = 2f(x).\ f(-x)$ ....(3)

(3) – (2)

$\Rightarrow \quad 2(f(x))^2 = 2f(x)f(-x)$

$\Rightarrow \quad f(x) = 0$ or $f(x) = f(-x)$

in both cases, function is an even function

H-6. $L_1 = \lim\limits_{x \to \infty} (f'(x) - \lambda f(x))$

$\Rightarrow \quad L_1 = \lim\limits_{x \to \infty} \dfrac{e^{-\lambda x}\{f'(x) - \lambda f(x)\}}{e^{-\lambda x}}$ [As it is a bounded function]

$\Rightarrow \quad L_1 = \lim\limits_{x \to \infty} \dfrac{e^{-\lambda x} f'(x) - \lambda f(x) e^{-\lambda x}}{e^{-\lambda x}}$

$\Rightarrow \quad L = L_1 = \lim\limits_{x \to \infty} \dfrac{\dfrac{d}{dx}\{f(x)e^{-\lambda x}\}}{\dfrac{d}{dx}\left(-e^{-\lambda x}.\dfrac{1}{\lambda}\right)}$

$\Rightarrow \quad L = \lim\limits_{x \to \infty} \dfrac{f(x)e^{-\lambda x}}{\left(-\dfrac{e^{-\lambda x}}{\lambda}\right)} \Rightarrow L = \lim\limits_{x \to \infty} -\lambda f(x)$

$\Rightarrow \quad L = -\lambda \lim\limits_{x \to \infty} f(x)$

$\Rightarrow \quad L = -\lambda L_2 \Rightarrow L_2 = -\dfrac{L}{\lambda}$

H-7. $f : R \to R \quad |f(x) - f(y)| \le |x - y|^3 \quad \forall\, x, y \in R \quad \left|\dfrac{f(x) - f(x)}{x - y}\right| \le |x - y|^2$

taking lim $x \to y$ on both side we have $|f'(y)| \le 0$

$\Rightarrow f'(y) = 0$

$\Rightarrow f(y)$ is constant $\forall\ y \in R$

PART - II

Section (A)

A-1. $\lim\limits_{x\to 0^+} \sec x > 1$ So ℓimit not exist

A-2. **S_1 :** False as Nr. is exactly zero as $x \to 0^+$

A-3. $\lim\limits_{x\to 1} (1 - x + [x - 1] + [1 - x])$

L.H.L.= $\lim\limits_{x\to 1^-} (1 - x + [x - 1] + [1 - x]) = \lim\limits_{h\to 0} (1 - (1-h) + [1 - h - 1] + [1 - 1 + h])$

$\lim\limits_{h\to 0} = (h + [-h] + [h]) = 0 - 1 + 0 = -1$

R.H.L. $\lim\limits_{h\to 0} =(1 - x + [x - 1] + [1 - x]) = (1 - (1+ h) + [1 + h - 1] + [1 - (1 + h)]$

$= \lim\limits_{h\to 0} (- h + [h] + [-h]) = 0 + 0 - 1 = -1$

L.H.L. = R.H.L. = – 1 so $\lim\limits_{x\to 1} (1 - x + [x - 1] + [1 - x]) = -1$

A4. $\lim\limits_{x\to 0^-} \dfrac{-x}{x} = -1$

Section (B)

B-1. $\lim\limits_{x\to 3} \dfrac{(x^3+27)\ \ell n\ (x-2)}{(x^2-9)} = \lim\limits_{x\to 3} \dfrac{(x+3)(x^2-3x+9)\ \ \ell n[1+(x-3)]}{(x-3)(x+3)} = \lim\limits_{x\to 3} (x^2 - 3x + 9).\ \dfrac{\ell n[1+(x-3)]}{(x-3)}$

(9 – 9 + 9) (1) = 9

B-2. $\lim\limits_{x\to 0} \dfrac{(4^x-1)^3}{\sin\left(\frac{x}{p}\right)\ \ell n\left(1+\frac{x^2}{3}\right)} = \lim\limits_{x\to 0} \dfrac{x^3\left(\frac{4^x-1}{x}\right)^3}{\left(\frac{x}{p}\right)\left[\dfrac{\sin\left(\frac{x}{p}\right)}{\frac{x}{p}}\right].\ell n\left(1+\frac{x^2}{3}\right)}$

$= 3p.\ \lim\limits_{x\to 0} \dfrac{\left(\frac{4^x-1}{x}\right)^3}{\left[\dfrac{\sin\left(\frac{x}{p}\right)}{\frac{x}{p}}\right]}\ .\ \dfrac{\left(\frac{x^2}{3}\right)}{\ell n\left(1+\frac{x^2}{3}\right)} = 3\,p\,(\ell n\ 4)^3$

B-3. Let x = 2 + h $x \to 2$, $h \to 0$ = $\lim\limits_{h\to 0} \dfrac{\sin(e^h-1)}{\ell n\ (1+h)} = \lim\limits_{h\to 0} \dfrac{\sin(e^h-1)}{(e^h-1)} \times \dfrac{(e^h-1)}{h} \times \dfrac{1}{\dfrac{\ell n\ (1+h)}{h}}$ = 1. 1. 1 = 1

B-4. $\lim\limits_{x\to 0} \dfrac{\sin(\ell n(1+x))}{\ell n(1+\sin x)} = \lim\limits_{x\to 0} \dfrac{\sin(\ell n(1+x))}{\ell n(1+x)}.\dfrac{\sin x}{\ell n(1+\sin x)}.\dfrac{\ell n(1+x)}{x}.\dfrac{x}{\sin x}$ = 1 .1 .1 .1 = 1

B-5. $\lim\limits_{x\to 1} \dfrac{\sqrt{1-\cos 2(x-1)}}{x-1}$; $\lim\limits_{x\to 1} \dfrac{\sqrt{2\sin^2(x-1)}}{x-1}$

$\lim\limits_{x\to 1} \dfrac{\sqrt{2}\,|\sin(x-1)|}{x-1}$; LHL = $\lim\limits_{h\to 0} \dfrac{\sqrt{2}\ \sin\ h}{-h} = -\sqrt{2}$ RHL = $\lim\limits_{h\to 0} \dfrac{\sqrt{2}\ \sin\ h}{h} = \sqrt{2}$

$\because$ L.H.L. $\neq$ R.H.L. So limit does not exist.

B-6. $\lim\limits_{x\to 0} \dfrac{(1+x^2)^{1/3}-(1-2x)^{\frac{1}{4}}}{x+x^2} = \lim\limits_{x\to 0} \dfrac{\left(1+\dfrac{x^2}{3}+....\right) - \left(1-\dfrac{x}{2}+\dfrac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}4x^2+....\right)}{x+x^2} = \dfrac{1}{2}$

B-7. $\lim\limits_{x\to 0} = \dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x} = \lim\limits_{x\to 0} \dfrac{2\sqrt{2}\sin^2\frac{x}{4}}{\sin^2 x} = \lim\limits_{x\to 0} \dfrac{2\sqrt{2}\ \sin^2\frac{x}{4}}{16.\ \dfrac{x^2}{16}.\ \dfrac{\sin^2 x}{x^2}} = \dfrac{\sqrt{2}}{8}$

B-8. $\lim\limits_{x\to 0^+} \dfrac{\cos^{-1}(1-x)}{\sqrt{x}} = \lim\limits_{h\to 0} \dfrac{\cos^{-1}[1-(0+h)]}{\sqrt{0+h}} = \lim\limits_{h\to 0} \dfrac{\cos^{-1}(1-h)}{\sqrt{h}}$

Let $1-h = \cos\theta$

$\sin\theta = \sqrt{1-(1-h)^2}$

$\therefore \quad \theta = \sin^{-1}\sqrt{2h-h^2} = \lim\limits_{h\to 0} \dfrac{\sin^{-1}\sqrt{2h-h^2}}{\sqrt{h}} = \lim\limits_{h\to 0} \dfrac{\sin^{-1}\sqrt{2h-h^2}}{\sqrt{2h-h^2}} \cdot \dfrac{\sqrt{2h-h^2}}{\sqrt{h}} = 1\times\sqrt{2} = \sqrt{2}$

B-9. $\lim\limits_{x\to 1} \dfrac{\left(\sum\limits_{k=1}^{100} x^k\right) - 100}{x-1} = \lim\limits_{x\to 1}\left[\dfrac{x-1}{x-1}+\dfrac{x^2-1}{x-1}+.....+\dfrac{x^{100}-1}{x-1}\right] = 1+2+3+.....+100 = \dfrac{(100).(101)}{2} = 5050$

B-10. $\lim\limits_{x\to\infty} \dfrac{x^3\ \sin\frac{1}{x} + x + 1}{x^2+x+1} = \lim\limits_{x\to\infty} \dfrac{\dfrac{\sin\frac{1}{x}}{\frac{1}{x}} + \dfrac{1}{x} + \dfrac{1}{x^2}}{1+\dfrac{1}{x}+\dfrac{1}{x^2}} = \dfrac{1+0+0}{1+0+0} = 1$

B-11. $\lim\limits_{x\to -\infty} \dfrac{x^2\ \sin\left(\frac{1}{x}\right)}{\sqrt{9x^2+x+1}} = \lim\limits_{x\to\infty} \dfrac{-x^2\ \sin\frac{1}{x}}{\sqrt{9x^2-x+1}} = \lim\limits_{x\to\infty} \dfrac{-\left(\dfrac{\sin\frac{1}{x}}{\frac{1}{x}}\right)}{\sqrt{9-\dfrac{1}{x}+\dfrac{1}{x^2}}} = -\dfrac{1}{3}$

B-12. $\lim\limits_{n\to\infty} \dfrac{5^{n+1}+3^n-2^{2n}}{5^n+2^n+3^{2n+3}} = \lim\limits_{n\to\infty} \dfrac{5.5^n+3^n-4^n}{5^n+2^n+27.9^n} = \lim\limits_{n\to\infty} \dfrac{5\ .\left(\frac{5}{9}\right)^n+\left(\frac{3}{9}\right)^n-\left(\frac{4}{9}\right)^n}{\left(\frac{5}{9}\right)^n+\left(\frac{2}{9}\right)^n+27} = \dfrac{0+0-0}{0+0+27} = 0$

B-13. $\lim\limits_{n\to\infty} n\cos\left(\dfrac{\pi}{4n}\right)\sin\left(\dfrac{\pi}{4n}\right) = \lim\limits_{n\to\infty}\cos\left(\dfrac{\pi}{4n}\right)\lim\limits_{n\to\infty}\dfrac{\sin\left(\frac{\pi}{4n}\right)}{\left(\frac{\pi}{4n}\right)}\times\dfrac{\pi}{4} = 1\ \dfrac{\pi}{4}.\ 1. = \dfrac{\pi}{4}$

B-14. $\sin h < h < \tan h,\ h \in \left(0, \dfrac{\pi}{2}\right)\ \dfrac{h}{\sin h} > 1 \Rightarrow \dfrac{-h}{\sin h} < -1$ LHL $= \lim\limits_{h\to 0}\left[\dfrac{\frac{\pi}{2}-h-\frac{\pi}{2}}{\cos\left(\frac{\pi}{2}-h\right)}\right] = \lim\limits_{h\to 0}\left[\dfrac{-h}{\sin h}\right] = -2$

RHL $= \lim\limits_{h\to 0}\left[\dfrac{\frac{\pi}{2}+h-\frac{\pi}{2}}{\cos\left(\frac{\pi}{2}+h\right)}\right] = \lim\limits_{h\to 0}\left[\dfrac{-h}{\sin h}\right] = -2$

$\therefore$ LHL = RHL = -2

B-15. $\lim\limits_{n\to\infty} \dfrac{-3n+(-1)^n}{4n-(-1)^n} = \lim\limits_{n\to\infty} \dfrac{-3+(-1)^n \frac{1}{n}}{4-(-1)^n \cdot \frac{1}{n}} = \dfrac{-3+0}{4-0} = -\dfrac{3}{4}$

B16. $\lim\limits_{x\to 1}\left(\dfrac{2}{1-x^2}+\dfrac{1}{x-1}\right) = \lim\limits_{x\to 1}\left(\dfrac{1}{1-x}+\dfrac{1}{1+x}+\dfrac{1}{x-1}\right) = \lim\limits_{x\to 1}\left(\dfrac{1}{x+1}\right) = \dfrac{1}{2}$

B-17. $\lim\limits_{x\to\infty}\left(x-x^2\ \ell n\left(1+\dfrac{1}{x}\right)\right) = \lim\limits_{x\to\infty} x - x^2\left(\dfrac{1}{x}-\dfrac{1}{2x^2}+\dfrac{1}{3x^3}-......\right) = \lim\limits_{x\to\infty} x - x + \dfrac{1}{2} - \dfrac{1}{3x} + ... = \dfrac{1}{2}$

B-18. $\lim\limits_{x\to 0} \dfrac{e^{-\frac{x^2}{2}} - \cos x}{x^3 \sin x} = \lim\limits_{x\to 0} \dfrac{\left[1-\dfrac{x^2}{2}+\dfrac{x^4}{4\ .\ 2!}-......\right]-\left[1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-......\right]}{x^3\left[x-\dfrac{x^3}{3!}+.......\right]}$

$= \lim\limits_{x\to 0} \dfrac{x^4\left[\dfrac{1}{8}-\dfrac{1}{4\ !}\right]-x^6\left[\dfrac{1}{8.\ 3!}-\dfrac{1}{6\ !}\right]+.....}{x^4\left[1-\dfrac{x^2}{3!}+.......\right]} = \left[\dfrac{1}{8}-\dfrac{1}{24}\right] = \dfrac{1}{12}$

B-19. $\lim\limits_{x\to 0} \dfrac{\sin(6x^2)}{\ell n\cos(2x^2-x)} = \lim\limits_{x\to 0} \dfrac{\sin 6x^2}{6x^2}.\dfrac{6x^2}{\ell n(\cos(2x^2-x))}$

$= 1.\ \lim\limits_{x\to 0} \dfrac{6x^2}{\ell n(\cos(2x^2-x))}\ \left(\dfrac{0}{0}\ \text{form}\right)$

$= \lim\limits_{x\to 0} \dfrac{12x}{\dfrac{(-\sin\ (2\ x^2\ -\ x))\ (4x-1)}{\cos\ (2x^2-x)}} = \lim\limits_{x\to 0} \dfrac{-12\ \cos(2x^2-x)}{4x-1}\ .\ \lim\limits_{x\to 0} \dfrac{x}{\sin(2x^2-x)}$

$= \left(\dfrac{-12}{-1}\right)\ .\ \lim\limits_{x\to 0} \dfrac{1}{(\cos(2x^2-x))(4x-1)} = 12.\left(\dfrac{1}{-1}\right) = -12$

B-20. Required $= \left.\dfrac{d^3(\sin x)}{dx}\right|_{x=a} = -\cos x$ at $x = a$

$\Rightarrow \quad \lim\limits_{h\to 0} \dfrac{\sin(a+3h)-3\sin(a+2h)+3\sin(a+h)-\sin a}{h^3} = -\cos a$

Section (C)

C-1. $\lim\limits_{x\to\infty}\left(\dfrac{x+2}{x-2}\right)^{x+1}$ (1^∞ form) $= e^{\lim\limits_{x\to\infty}\left(\frac{x+2}{x-2}-1\right)\ .\ (x+1)} = e^{\lim\limits_{x\to\infty}\frac{4(x+1)}{(x-2)}} = e^{\lim\limits_{x\to\infty}\frac{4\left(1+\frac{1}{x}\right)}{\left(1-\frac{2}{x}\right)}} = e^4$

C-2. $\lim\limits_{x\to 0^+}\left(1+\tan^2\sqrt{x}\right)^{\frac{5}{x}} = \lim\limits_{h\to 0}\left(1+\tan^2\sqrt{0+h}\right)^{\frac{5}{0+h}} = \lim\limits_{h\to 0}\left(1+\tan^2\sqrt{h}\right)^{\frac{5}{h}}$ (1^∞ form)

$= e^{\lim\limits_{h\to 0}\frac{5\tan^2\sqrt{h}}{h}} = e^{\lim\limits_{h\to 0} 5\left(\frac{\tan\sqrt{h}}{\sqrt{h}}\right)^2} = e^5$

C-3. $\lim\limits_{x\to\frac{\pi}{4}}(1+[x])^{\frac{1}{\ell n\ (\tan x)}} = \lim\limits_{x\to\frac{\pi}{4}}(\text{exact}\ \ 1)^{\frac{1}{\ell n\ (\tan x)}} = 1$

C-4. $\lim\limits_{x\to\infty}\left(\dfrac{x^2-2x+1}{x^2-4x+2}\right)^x = e^{\lim\limits_{x\to\infty}\left(\frac{x^2-2x+1-x^2+4x-2}{x^2-4x+2}\right)x} = e^{\lim\limits_{x\to 0}\left(\frac{(2x-1)x}{x^2-4x+2}\right)} = e^{\lim\limits_{x\to\infty}\left(\frac{2-\frac{1}{x}}{1-\frac{4}{x}+\frac{2}{x^2}}\right)} = e^2$

C-5. $\lim\limits_{x\to 0} (\cos x)^{\frac{1}{\sin x}}$ (1^∞ form) = $e^{\lim\limits_{x\to0}\left(\frac{\cos x-1}{\sin x}\right)}$ = $e^{\lim\limits_{x\to0}\frac{-2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}}$ = $e^{\lim\limits_{x\to0}-\tan\left(\frac{x}{2}\right)}$ = e^0 =1

C-6. $\lim\limits_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$ (1^∞ form)

$= e^{\lim\limits_{x\to a}\left(2-\frac{a}{x}-1\right).\tan\left(\frac{\pi x}{2a}\right)} = e^{\lim\limits_{x\to a}\left(\frac{x-a}{x}\right)\tan\left(\frac{\pi x}{2a}\right)} = e^{\frac{1}{a}\lim\limits_{x\to a}\frac{x-a}{\cot\left(\frac{\pi x}{2a}\right)}} = e^{\frac{1}{a}\lim\limits_{x\to a}\frac{1}{-\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)}\times\frac{2a}{\pi}} = e^{\frac{1}{a}\cdot\left(\frac{-2a}{\pi}\right)} = e^{-\frac{2}{\pi}}$

C-7. Required = $e^{\lim\limits_{t\to0}\left(\frac{\cos tx-1}{t^2}\right)}$ = $e^{\lim\limits_{t\to0}-x^2\left(\frac{1-\cos tx}{t^2x^2}\right)}$ = $e^{-\frac{1}{2}x^2}$

C-8. $\lim\limits_{n\to\infty}\frac{1}{n^4}\left([1^3x]+[2^3x]+\ldots+[n^3x]\right)$

$1^3x-1<[1^3x]\ 1^3x$
$2^3x-1<[2^3x]\ 2^3x$
. . .
. . .
. . .
$n^3x-1<[n^3x]\ n^3x$

Adding all these inequilities

$(1^3+2^3+3^3\ldots\ldots\ldots..+n^3)\,x-n<[1^3x]+[2^3x]+\ldots\ldots\ldots..[n^3x]\le(1^3+2^3+\ldots\ldots\ldots.n^3)\,x$

$$\frac{\frac{n^2(n+1)^2}{4}x-n}{n^4}<\frac{[1^3x]+[2^3x]+\ldots.+[n^3x]}{n^4}\le\frac{\left(\frac{n(n+1)}{2}\right)^2x}{n^4}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\frac{x}{4}-\frac{1}{n^3}<\lim_{n\to\infty}\frac{[1^3x]+[2^3x]+\ldots\ldots..+[n^3x]}{n^4}\le\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\frac{x}{4}$$

$$\frac{x}{4}<\lim_{x\to\infty}\frac{[1^3x]+[2^3x]+\ldots\ldots.[n^3x]}{n^4}\le\frac{x}{4}$$

$$\therefore\quad\lim_{x\to\infty}\frac{[1^3x]+[2^3x]+\ldots\ldots.+[n^3x]}{n^4}=\frac{x}{4}$$

Section (D)

D-1. $\lim\limits_{x\to0}f(x)=f(0)$

$$\lim_{x\to0}\frac{\cos(\sin x)-\cos x}{x^2}=a\Rightarrow\frac{2}{x^2}\sin\left(\frac{\sin x+x}{2}\right)\sin\left(\frac{x-\sin x}{2}\right)=a$$

$$\Rightarrow a=\lim_{x\to0}\ 2.\frac{\sin\left(\frac{\sin x+x}{2}\right)}{\frac{\sin x+x}{2}}.\frac{\sin\left(\frac{x-\sin x}{2}\right)}{\frac{x-\sin x}{2}}.\frac{1}{4}\left(\frac{\sin x+x}{x}\right)\left(\frac{x-\sin x}{x}\right)=2.1.1\ \frac{1}{4}\ (1+1)\ (1-1)=0$$

D-2. $$f(x)=\left|\left(x+\frac{1}{2}\right)[x]\right|=\begin{cases}-(2x+1), & -2\le x<-1\\ |x+1/2|, & -1\le x<0\\ 0, & 0\le x<1\\ x+1/2, & 1\le x<2\\ 5, & x=2\end{cases}$$

f(x) is discontinous at x = –1, 0, $\frac{1}{2}$, 1, 2

D-3. $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{h\to 0} f(1-h) = \lim\limits_{h\to 0} \dfrac{\ell n\ [(1-h)^2 - 2(1-h) + 5]}{\ell n\ [1-4h]}$

does not exist as denominater is tending to zero. similarly $\lim\limits_{n\to 1^+}$ f(x) also does not exist.

D-4. $\lim\limits_{x\to 1^+} x\sin\left(\dfrac{\pi}{2}(x+2)\right) = \lim\limits_{x\to 1^+}(-x)\sin\left(\dfrac{\pi x}{2}\right) = -1$ and $\lim\limits_{x\to 1^-} x\sin\left(\dfrac{\pi}{2}(x+0)\right) = 1$

Section (E)

E-1. LHL (x = 0) = f(0) = RHL (x=0); LHL = $\text{Lim}_{x\to 0^-} \dfrac{\sqrt{1+P(x)} - \sqrt{1-Px}}{x} = \dfrac{2P}{2} = p$ $f(0) = -\dfrac{1}{2}$ = RHL

E-2. If [g(x)] = sgn(g(x)) = sgn (x(x² – 5x + 6)) = sgn (x (x – 2) (x – 3)) = $\begin{bmatrix} 1 & ; & x>3,\ 1<x<2 \\ 0 & ; & x=1,\ 2,\ 3 \\ -1 & ; & 2<x<3\ ;\ x<1 \end{bmatrix}$

f(g(x)) is discontinuous at 3 points (0, 2 and 3)

E-3. $y = \dfrac{1}{t^2+t-2}$, where $t = \dfrac{1}{x-1}$, y = f(x) is discontinuous at x = 1, where t is discontinuous and

$y = \dfrac{1}{(t+2)(t-1)}$ at t = – 2 and t = 1 $\Rightarrow \dfrac{1}{x-1} \Rightarrow -2x + 2 = 1,\ x = \dfrac{1}{2}$

$1 - \dfrac{1}{x-1} \qquad \Rightarrow \qquad x = 2$

f(g(x)) is discontinuous at $x = \dfrac{1}{2}$, 2,1

E-4. Let f(x) = 2 tan x + 5x – 2

$f(0) = -2 \qquad f(\pi/4) = 2\tan\dfrac{\pi}{4} + \dfrac{5\pi}{4} - 2 = \dfrac{5\pi}{4}$

Now $x \in \left[-2, \dfrac{5\pi}{4}\right]$ and f(x) is continuous on [0, π/4]

∴ By intermediate value theorem c ∈ [0, π/4] for which f(c) = 0

∴ (b) is correct.

Section (F)

F-1. $f(x) = x\left(\sqrt{x} - \sqrt{x+1}\right)$ is continuous at x = 0

$f'(x) = x\left(\sqrt{x} - \sqrt{x+1}\right) + x \left(\dfrac{1}{2\sqrt{x}} - \dfrac{1}{2\sqrt{x+1}}\right)$

$f'(x) = \sqrt{x} - \sqrt{x+1} + \dfrac{\sqrt{2}}{2} - \dfrac{x}{2\sqrt{x+1}}$

f '(0⁺) f '(0⁻) = –1

Hence it is differentiable at x = 0

F-2. $f(x) = \begin{cases} \dfrac{x(3e^{1/x}+4)}{2-e^{1/x}}, & x \neq 0 \\ 0 & x = 0 \end{cases}$

$f'(0^-) = \lim\limits_{h\to 0} \dfrac{\dfrac{-h(3e^{-1/h}+4)}{2-e^{-1/h}} - 0}{-h} = 2$

$f'(0^+) = \lim\limits_{x\to 0} f(x) = 0 = \lim\limits_{h\to 0^+} f(x) = f(0)$ not differentiable continuous

F-3. $f(x) = \dfrac{x}{\sqrt{x+1}-\sqrt{x}}$ $\qquad$ $f(0) = 0$ $\qquad$ Domain $x \geq 0$

$\because$ $\lim\limits_{x \to 0^+} f(x) = 0$

$\therefore$ f(x) is continous

$\because$ RHD (x = 0) = $\lim\limits_{h \to 0} \dfrac{\dfrac{h}{\sqrt{h+1}-\sqrt{h}} - 0}{h} = 1$

$\therefore$ f(x) is differntiable at x = 0

F-4. $f(x) = \sin^{-1}(\cos x) = \dfrac{\pi}{2} - \cos^{-1}\cos x = \begin{cases} \dfrac{\pi}{2} + x & , \quad x \in [-\pi, \ 0] \\ \dfrac{\pi}{2} - x & , \quad x \in [0, \ \pi] \end{cases}$

continous but not differentiable at x =0

F-5. f (0) = $\lim\limits_{x \to 0} f(x) = 0 - 1 + 0.\sin(-1) = -1$

f (0$^+$) = $\lim\limits_{x \to 0} f(x) = 0 + 0 + 0.\sin 0 = 0$ = f(0)

f(x) is not continuous at x = 0 at x = 2,

$f(2^+) = 2 + 2 + 2 \sin 2 = 4 + 2 \sin 2$

$f(2^-) = 2 + 1 + 2 \sin 1 = 3 + 2 \sin 1$

f(x) is not continuous at x = 2

F-6. Obvious

F-7. $\lim\limits_{x \to 2^-} f(x) = \dfrac{3}{5} = f(2) \neq \lim\limits_{x \to 2^+} f(x) = 1$

f(x) is not continous at x = 2

$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} f(x) = f(3) = \dfrac{9}{2}$

Now LHD (x =3) is $\lim\limits_{h \to 0} \dfrac{\dfrac{1}{4}((3-h)^3 - (3-h)^2) - \dfrac{9}{2}}{-h} = \lim\limits_{h \to 0} \dfrac{h^2 - 8 \ \ h + 21}{4} = \dfrac{21}{4}$

and RHD (x = 3) is $\lim\limits_{h \to 0} \dfrac{\dfrac{9}{4}(|h-1| + |-1-h|) - \dfrac{9}{2}}{h} = 0$

Section (G)

G-1. $f(x) = \begin{cases} \dfrac{x}{1-x} & , \quad x < 0 \\ \dfrac{x}{1+x} & , \quad x \geq 0 \end{cases}$ f(x) is continous and differntiable for $x \in R$

$f(x) = \begin{cases} \dfrac{x}{1-x} & , \quad x < 0 \\ \dfrac{x}{1+x} & , \quad x \geq 0 \end{cases}$ f(x) is discontinous at |x| = 1

G-2. If 'f' is differentiable then |f| is differentiable at each point x, where $f(x) \neq 0$

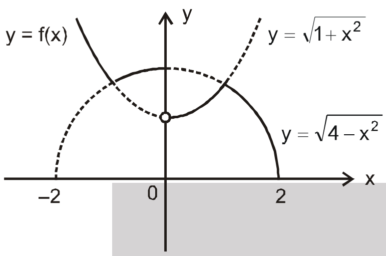
if $f(\alpha) = 0$ and $f'(\alpha) = 0$, then |f| is differentiable at $x = \alpha$ $\Rightarrow$ if $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then |f| is not differentiable at $x = \alpha$ $\Rightarrow$ If f is differentiable then |f| may or may not be differentiable,

[option A, C, D not necessarly true]

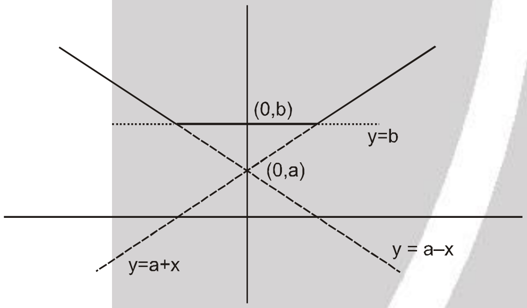
Now $|f|^2 = f^2 (f^2)' = 2.f.f'$ since f is differentiable

$\therefore$ f^2 is also differentiable

G-3.

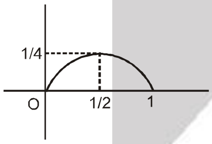


G-4. y = f(x) = max {a–x, a+x, b}, 0 < a < b

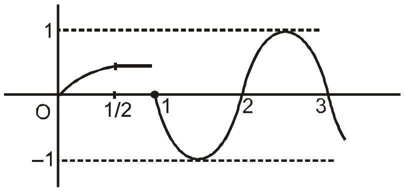


f(x) is non-differentiable at 2 points

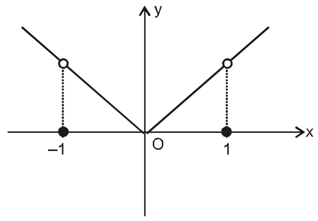
G-5.



$$y = g(x) = \begin{cases} x - x^2 & 0 \le x \le 1/2 \\ 1/4 & 1/2 < x \le 1 \\ \sin \pi x & x > 1 \end{cases}$$



G-6. **S_1:** $f(x) = |x \operatorname{sgn}(1 - x^2)| = \begin{cases} -x & x \in (-\infty, -1) \cup (-1, \infty) \\ 0 & x = -1, 0, 1 \\ x & x \in (0, 1) \cup (1, \infty) \end{cases}$



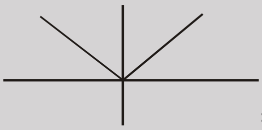
function is discontinous at x = –1, 1 and non differentiable at x = –1, 0,1

S_2 : $f(x) = a \sin \frac{\pi}{2} (x + 1)$, x 0 $= \frac{\tan x - \sin x}{x^3}$, $x > 0$

$a = \lim_{x \to 0^+} \frac{\tan x - \sin x}{x^3} = \lim_{x \to 0^+} \frac{\tan x(1 - \cos x)}{x^3} = \frac{1}{2}$

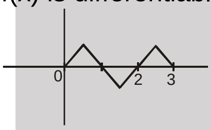
$\therefore \quad a = \frac{1}{2}$

S_3: $f(x) = (x^2 |x|)^{1/3} = \begin{cases} (x^3)^{\frac{1}{3}} = x & , x \geq 0 \\ (-x^5)^{\frac{1}{3}} = (-1)^{\frac{1}{3}} x = -x & , x < 0 \end{cases}$



= – x

f(x) is differentiable every where except at x = 0



S_4 :

f(x) will be non differentiable if $\sin^{-1}(\sin x) = 0$ or graph of f(x) has a sharp point. Hence number of points of non differentiable will be 5.

G-7. **S_1 :** $f(x) = \frac{\sin(\pi [x - \pi])}{1 + [x]^2}$ $[x - \pi]$ is an integer for $x \in R$

$\therefore$ f(x) = 0 $\forall$ $x \in R$.

Hence f(x) is always continuous. **(False)**

S_2 : f(x) = p[x + 1] + q [x – 1] = (p + q) [x] + p – q

f(1) = 2p

$f(1^+) = 2p$

$f(1^-) = p - q$

But f(x) is continous at x = 1

2p = p – q   p + q = 0 **[True]**

S_3 : $f(x) = |[x]\, x| = \begin{cases} -x & , -1 \leq x < 0 \\ 0 & , 0 \leq x < 1 \\ x & , 1 \leq x < 2 \\ 4 & , x = 2 \end{cases}$

$\because$ function is not continuous at x = 2

$\therefore$ non-differentiable also **(True)**

S_4 : f(0) = constant $f'(0) = 0\ \forall\ x \in R$

$f'(10) = 0$ [False]

G-8. $f(x) = \begin{cases} x & , \quad x \le 1 \\ ax^2 + bx + c & , \quad \text{otherwise} \end{cases}$ f(x) should be continous at x =1

it gives a+b+c =1

f(x) should be differentiable at x= 1

it gives 2a+b=1 $\Rightarrow$ b =1–2a c= 1–a–b= a

Section (H)

H-1. $\lim\limits_{h\to 0} \dfrac{6}{4} = \lim\limits_{h\to 0} \dfrac{f(2h+2+h^2)-f(2)}{(2h+2+h^2)-(2)} \; \dfrac{(h-h^2+1)-(1)}{f(h-h^2+1)-f(1)} \dfrac{(2h+2+h^2)-(2)}{(h-h^2+1)-(1)} = \lim\limits_{h\to 0} \dfrac{f'(2)}{f'(1)} . \dfrac{h(2+h)}{h(1-h)} = 3$

H-2. f(x + y) = f(x).f(y) and f(1) = 2

$\sum\limits_{n=1}^{10} f(n) = f(1) + f(2) + \ldots\ldots\ldots.. + f(10) = 2^1 + 2^2 + 2^3 + \ldots\ldots. + 2^{10} = 2\left(\dfrac{2^{10}-1}{2-1}\right) = 2046$

H-3. f(1) = 1 = 2 – 1

f(n + 1) = 2f(n) + 1

$\therefore$ f(2) = 2f(1) + 1 = 2. 1 + 1 = 3 = 2^2 – 1 $\Rightarrow$ f(3) = 7 = 2^3 – 1

f(4) = 15 = 2^4 – 1

Similarly f(n) = 2^n – 1

H-4. $f\left(x + \dfrac{1}{x}\right) = x^2 + \dfrac{1}{x^2} = \left(x + \dfrac{1}{x}\right)^2 - 2$

Replace $x + \dfrac{1}{x} = t$, where $|t| \ge 2$

$\therefore$ $f(t) = t^2 - 2$, $|t| \ge 2$

H-5. **Method 1 :** (usual but lengthy)

$x^2 f(x) + f(1 - x) = 2x - x^4$ .....(1)

replace x by (1 – x) in equation (1)

$(1 - x)^2 f(1 - x) + f(x) = 2(1 - x) - (1 - x)^4$ .....(2)

eliminate f(1 – x) by equation (1) and (2) we get $f(x) = 1 - x^2$

Method 2: Since R.H.S. is polynomial of 4th degree and also by options consider $f(x) = ax^2 + bx + c$

$x^2 f(x) + f(1 - x) = 2x - x^4 \Rightarrow x^2 (ax^2 + bx + c) + a(1 - x)^2 + b(1 - x) + c = 2x - x^4$

by comparing coefficients

a = – 1

b = 0

c = 1

$\therefore$ $f(x) = -x^2 + 1$

H-6. $f(x + 2y) = f(x) + f(2y) + 4xy \quad \forall \quad x, y \in R$

Replace 2y with y we have

$f(x + y) = f(x) + f(y) + 2xy \quad \forall \quad x, y \in R$

diff. w.r.t. x

f '(x+y) = f '(x) + 2y

Put x = 1 y = –1 f '(0) = f '(1) –2

PART - III

1. Consider the graph of 2 cos x in $(-\pi, \pi)$. 2 cos x is integer at 9 points.

[2 cos x] is discontinuous at 7 points in $(-\pi, \pi)$

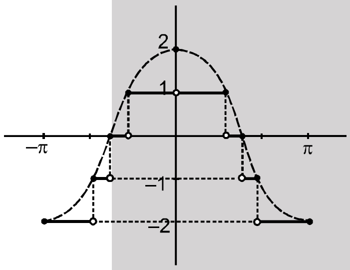
Similarly from graph of 2sinx, we can observe that [2 sin x] is discontinuous at 7 points

(continuous at $-\pi/2, \pi$)

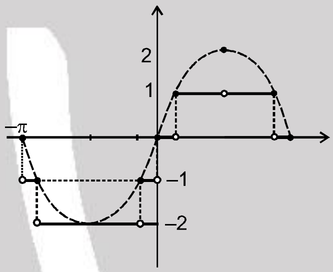
[2 tan x/2] is discontinuous at 4 points (continuous at $-\pi/2$)

[3 cosec x/3] is discontinuous at 4 points (continuous at $\pi/2$)

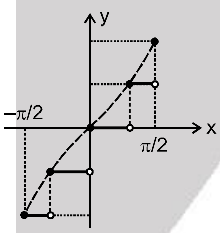
y = [2 cos x]



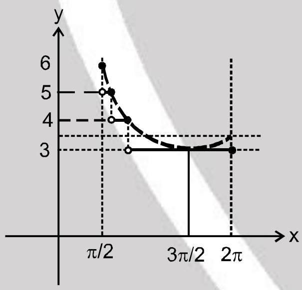
y = [2 sin x]



y = [2 tan x/2]



y = [3 cosec x/3]



2. (A) $f(x) = |x^3|$ is continuous and differentiable

(B) $f(x) = \sqrt{|x|}$ is continuous

$f'(x) = \dfrac{1}{2\sqrt{|x|}} \cdot \dfrac{x}{|x|}$ {does not exist at x = 0}

(C) $f(x) = |\sin^{-1} x|$ is continuous

$f'(x) = \dfrac{\sin^{-1} x}{|\sin^{-1} x|} \cdot \dfrac{1}{\sqrt{1-x^2}}$ {does not exist at x = 0}

(D) $f(x) = \cos^{-1} |x|$ is continuous

$f'(x) = \dfrac{-1}{\sqrt{1-x^2}} \cdot \dfrac{x}{|x|}$ {does not exist at x = 0}

EXERCISE # 2

PART–1

1. $\lim\limits_{x\to a^-}\left(\dfrac{|x|^3}{a}-\left[\dfrac{x}{a}\right]^3\right)$ (a > 0)

$x = a - h = \lim\limits_{h\to 0}\left(\dfrac{|a-h|^3}{a}-\left[\dfrac{a-h}{a}\right]^3\right) = \lim\limits_{h\to 0}\left(\dfrac{|a|^3}{a}-\left[1-\dfrac{h}{a}\right]^3\right) = a^2 - 0 = a^2$

2. $\lim\limits_{n\to\infty}\cos\dfrac{x}{2}\cos\dfrac{x}{2^2}\cos\dfrac{x}{2^3}\cos\dfrac{x}{2^4}.......\cos\dfrac{x}{2^n} = \lim\limits_{n\to\infty}\dfrac{\sin x}{2^n\sin\dfrac{x}{2^n}} = (\because \lim\limits_{n\to\infty}\dfrac{x}{2^n} = 0) = \dfrac{\sin x}{x}$

3. $\sin\theta < \theta < \tan\theta,\ \theta \in \left(0, \dfrac{\pi}{2}\right)\ \dfrac{\sin\theta}{\theta} < 1 < \dfrac{\tan\theta}{\theta}$

$\dfrac{n\sin\theta}{\theta} < n < \dfrac{n\tan\theta}{\theta}\quad \lim\limits_{\theta\to 0}\left(\left[\dfrac{n\sin\theta}{\theta}\right]+\left[\dfrac{n\tan\theta}{\theta}\right]\right)$; $n \in N$

L.H.L. $= \lim\limits_{\theta\to 0^-}\left(\left[\dfrac{n\sin\theta}{\theta}\right]+\left[\dfrac{n\tan\theta}{\theta}\right]\right) = n - 1 + n = 2n - 1$

R.H.L. $= \lim\limits_{\theta\to 0^+}\left(\left[\dfrac{n\sin\theta}{\theta}\right]+\left[\dfrac{n\tan\theta}{\theta}\right]\right) = n - 1 + n = 2n - 1$

$\therefore$ L.H.L. = R.H.L = 2n – 1

4. R.H.L. $= \lim\limits_{x\to 0^+}\left[(1-e^x)\dfrac{\sin x}{x}\right]$ when $x \in (0, h)$ and $h \to 0$ then $(1 - e^x) \in (-1, 0)$ and $\dfrac{\sin x}{x} < 1$

So $-1 < (1 - e^x)\ \dfrac{\sin x}{x} < 0$; $\lim\limits_{x\to 0^+}\left[(1-e^x)\dfrac{\sin x}{x}\right] = -1$

L.H.L. $= \lim\limits_{x\to 0^-}\left[(1-e^x)\dfrac{\sin x}{-x}\right] = \lim\limits_{x\to 0^-}\left[(e^x-1)\dfrac{\sin x}{x}\right]$

when $x \in (-h, 0)$ and $h \to 0$, then $e^x - 1 \in (-1, 0)$ and $\dfrac{\sin x}{x} < 1$

So $-1 < (e^x - 1)\ \dfrac{\sin x}{x} < 0$

So $\lim\limits_{x\to 0^-}\left[(e^x-1)\dfrac{\sin x}{x}\right] = -1$ L.H.L. = R.H.L. = –1

5. $\lim\limits_{x\to 0}\dfrac{\cos(\sin x)-\cos x}{x^4} = \lim\limits_{x\to 0}\dfrac{\cos\left(x-\dfrac{x^3}{3!}+...\right)-\cos x}{x^4}$

$$= \lim\limits_{x\to 0}\frac{\left[1-\dfrac{\left(x-\dfrac{x^3}{6}+...\right)^2}{2!}+\dfrac{\left(x-\dfrac{x^3}{6}+...\right)^4}{4!}-......\right]-\left[1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-....\right]}{x^4}$$

$$= \lim\limits_{x\to 0}\frac{x^4\left(\dfrac{1}{2}.\dfrac{2}{6}+\dfrac{1}{24}\right)-\dfrac{x^4}{24}+......}{x^4} = \frac{1}{6}+\frac{1}{24}-\frac{1}{24} = \frac{1}{6}$$

6. $\lim\limits_{t\to 1}\dfrac{t-t^t}{1-t+\ell nt} = \lim\limits_{t\to 1}\dfrac{1-t^t(1+\ell nt)}{-1+\dfrac{1}{t}}$ (LH. rule) $= \lim\limits_{t\to 1}\dfrac{-t^{t-1}-t^t(1+\ell nt)^2}{-\dfrac{1}{t^2}}$ (LH. rule) $= \dfrac{-1-1}{-1} = 2$

7. $\lim\limits_{x \to \frac{\pi}{2}} \tan^2 x \left(\sqrt{2\sin^2 x + 3\sin x + 4} - \sqrt{\sin^2 x + 6\sin x + 2}\right)$

$= \lim\limits_{x \to \frac{\pi}{2}} \tan^2 x \left(\dfrac{2\sin^2 x + 3\sin x + 4 - (\sin^2 x + 6\sin x + 2)}{\sqrt{2\sin^2 x + 3\sin x + 4} + \sqrt{\sin^2 x + 6\sin x + 2}}\right)$

$= \lim\limits_{x \to \frac{\pi}{2}} \tan^2 x \dfrac{(\sin^2 x - 3\sin x + 2)}{\sqrt{2+3+4} + \sqrt{1+6+2}}$.

$= \lim\limits_{x \to \frac{\pi}{2}} \dfrac{1}{6}\left[\dfrac{\sin^2 x - 3\sin x + 2}{\cos^2 x}\right]$ $\left(\dfrac{0}{0} \text{ form}\right)\left(\dfrac{0}{0} \text{ रूप}\right)$ (use L'Hospital rule)

$= \dfrac{1}{6} \lim\limits_{x \to \frac{\pi}{2}} \dfrac{2\sin x \cos x - 3\cos x}{2\cos x(-\sin x)} = \dfrac{1}{6} \lim\limits_{x \to \frac{\pi}{2}} \dfrac{2\sin x - 3}{-2\sin x} = \left(\dfrac{1}{6}\right)\left(\dfrac{1}{2}\right) = \dfrac{1}{12}$

8. α, β are the roots of the equation $ax^2 + bx + c = 0$

$\because \quad ax^2 + bx + c = a(x-\alpha)\ (x-\beta)$

$\lim\limits_{x\to\alpha}\left(1 + ax^2 + bx + c\right)^{\frac{1}{x-\alpha}} = e^{\lim\limits_{x\to\alpha}\frac{ax^2+bx+c}{x-\alpha}} = e^{\lim\limits_{x\to\alpha}\frac{a(x-\alpha)\ (x-\beta)}{(x-\alpha)}} = e^{a(\alpha-\beta)}$

9. $\lim\limits_{x\to\infty} \dfrac{e^x\left(\left(2^{x^n}\right)^{\frac{1}{e^x}} - \left(3^{x^n}\right)^{\frac{1}{e^x}}\right)}{x^n}$, $n \in N$

$\lim\limits_{x\to\infty} \dfrac{(2)^{\frac{x^n}{e^x}} - (3)^{\frac{x^n}{e^x}}}{\frac{x^n}{e^x}}$ when $x \to \infty$, $\dfrac{x^n}{e^x} \to 0$

Put $\dfrac{x^n}{e^x} = t$

$\lim\limits_{t\to 0}\left(\dfrac{2^t - 3^t}{t}\right) = \ln 2 - \ln 3 = \ln\left(\dfrac{2}{3}\right)$

10. $\lim\limits_{y\to 0}\left(\lim\limits_{x\to\infty} \dfrac{\exp\left(x \ln\left(1 + \frac{ay}{x}\right)\right) - \exp\left(x \ln\left(1 + \frac{by}{x}\right)\right)}{y}\right)$

$= \lim\limits_{y\to 0}\left(\lim\limits_{x\to\infty} \dfrac{\left(1 + \frac{ay}{x}\right)^x - \left(1 + \frac{by}{x}\right)^x}{y}\right)$ by expansion

$= \lim\limits_{y\to 0}\left(\lim\limits_{x\to\infty} \dfrac{\left(1 + ay + \frac{x(x-1)}{2!}.\frac{a^2 y^2}{x^2} + \ldots\right) - \left(1 + by + \frac{x(x-1)}{2!}.\frac{b^2 y^2}{x^2} + \ldots\right)}{y}\right)$

$= \lim\limits_{y\to 0}\left[\dfrac{y(a-b) + \frac{y^2}{2}(a^2 - b^2) + \ldots}{y}\right] = a - b$

11. $f(x) = \lim_{t\to 0}\left(\frac{2x}{\pi}\cot^{-1}\frac{x}{t^2}\right)$

Case-I : when x = 0

$f(x) = 0$

Case-II : when x > 0

$f(x) = \lim_{t\to 0}\left(\frac{2x}{\pi}\cot^{-1}\frac{x}{t^2}\right) = \frac{2x}{\pi}\times\cot^{-1}(\infty) = .\ \frac{2x}{\pi}\ 0 = 0$

Case-III when x < 0

$f(x) = \lim_{t\to 0}\left(\frac{2x}{\pi}\cot^{-1}\frac{x}{t^2}\right) = \frac{2x}{\pi}\times\cot^{-1}(-\infty) = \frac{2x}{\pi}.\ \pi = 2x \Rightarrow f(x) = 2x$

12. $g(x) = x - [x] = \{x\} \in [0, 1)$

g(x) is discontinuous only at $x \in I$

Now $h(x) = fog(x)$

h(x) is continuous $\forall\ x \in R - I$

Let $x \in I$, consider x = n

$h(n) = f[g(n)] = f(0)$

$\lim_{x\to n^-} h(x) = \lim_{x\to n^-} f(\{x\}) = f(1) = f(0)$

$\lim_{x\to n^+} h(x) = \lim_{x\to n^+} f(\{x\}) = f(0) \Rightarrow$ h(x) is continuous $x \in I$

h(x) is continuous $\forall\ x \in R$

13. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{(a+b+5)+\left(-a-\frac{b}{2}\right)x^2+.....}{x^2} = 3$

$\Rightarrow$ $a + b + 5\ 0\ \&\ -a - \frac{b}{2} = 3 \Rightarrow a = -1, b = -4$

Because $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(1+\left(\frac{cx+dx^3}{x^2}\right)\right)^{1/x}$ exists so, $\lim_{x\to 0^+}\left(\frac{(x+dx)^3}{x^2}\right) = 0 \Rightarrow c = 0$

Now $\lim_{x\to 0^+}(1 + dx)^{1/x} = e^d = 3$

14. $f(x) = \begin{cases} x^2 & x \text{ is irational} \\ 1 & x \text{ is rational}\end{cases}$

It is continuous $x^2 = 1$ $x = \pm 1$

Discontinuous at all x except x = 1, –1

15. $f(x) = [\sin[x]] = \begin{cases} [\sin 0] = 0 & 0 \le x < 1 \\ [\sin 1] = 0 & 1 \le x < 2 \\ [\sin 2] = 0 & 2 \le x < 3 \\ [\sin 3] = 0 & 3 \le x < 4 \\ [\sin 4] = -1 & 4 \le x < 5 \\ [\sin 5] = -1 & 5 \le x < 6 \\ [\sin 6] = -1 & 6 \le x < 2\pi \end{cases}$

f(x) is discontinuous at (4, –1)

16. f(x) = $\begin{cases} 0 & x \in I \\ (-1)^n & x \in (n, n+1), \quad n \in I \end{cases}$

f(x) = discontinous at all integer value of x.

17. $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{h \to 0} -\sqrt{h}\left(1 - h\sin\left(-\frac{1}{h}\right)\right) = 0$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{h \to 0} \sqrt{h}\left(1 + h\sin\ \frac{1}{h}\right) = 0$

f(0) = 0

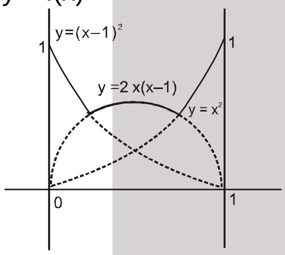
f(x) is continuous at x = 0

LHD at x = 0 is $\lim\limits_{h \to 0} \dfrac{-\sqrt{h}\ \left(1 - h\sin\ \left(-\frac{1}{h}\right)\right) - 0}{-h} = \lim\limits_{h \to 0} \dfrac{1 - h\sin\ \left(-\frac{1}{h}\right)}{\sqrt{h}} = \infty$

LHD is not define so f(x) is not differentiable at x = 0

18. y = f(x) = max {x^2, $(x-1)^2$, $2x(1-x)$}

y = f(x)

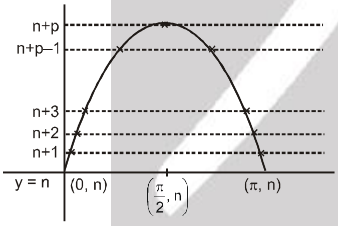


Non-differentiable at two points.

19. f(x) = [x] [sin πx], x ∈ (–1, 1) = $\begin{cases} 1\ ; & x \in (-1, 0) \\ 0\ ; & x \in [0, 1) \end{cases}$

f(x) is continuous in (–1, 0)

20. f(x) = [n + psinx], x ∈ (0, π) graph of y = n + p sinx



obviously

f(x) = [n +psinx] is discontinous at points mark in above curve

⇒ number of such points ⇒ (p –1) + 1 + p –1 = 2p –1

21. Given g(x) = $\dfrac{1}{f(x)}$ g(x) cannot be define where f(x) = 0

⇒ if f(x) is onto, g(x) may or may not be onto

same argument can be given for continuity and differentiability of g(x) if f(x) is one-one

⇒ for $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$g(x_1) \neq g(x_2)$

⇒ g(x) is also one-one

22. $f(x) = x^3 - x^2 + x + 1$

$f'(x) = 3x^2 - 2x + 1 > 1 \quad , \quad \forall \quad x \in R$

f(x) is strictly increasing

So $g(x) = \begin{cases} \max \{f(t) \ ; \ 0 \le t \le x\} \ , & 0 \le x \le 1 \\ 3 - x + x^2, & 1 < x \le 2 \end{cases} = \begin{cases} f(x) & 0 \le x \le 1 \\ 3 - x + x^2 & 1 < x \le 2 \end{cases}$

$g(1) = f(1) = 1 - 1 + 1 + 1 = 2 = \lim\limits_{x \to 1^-} g(x)$

$\lim\limits_{x \to 1^+} g(x) = 3 - 1 + 1 = 3$

g(x) is neither continous nor differentiable at x = 1

23. $f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}$ ....(1) $\qquad f(0) = 0, \ f'(0) = 3$

Put x = 3x and y = 0

$f(x) = \frac{f(3x)}{3}$ ....(2)

$\lim\limits_{h \to 0} f(x+h) = \lim\limits_{h \to 0} f\left(\frac{3x+3h}{3}\right) = \lim\limits_{h \to 0} \frac{f(3x)+f(3h)}{3} = \frac{f(3x)}{3} = f(x)$

Similarly we can prove $\lim\limits_{h \to 0} f(x-h) = f(x)$ f(x) is continuous for all x in R

Given that $f'(0) = 3 \Rightarrow \lim\limits_{h \to 0} \frac{f(h)}{h} = \lim\limits_{h \to 0} \frac{f(-h)}{-h} = 3$

24. $f\left(\frac{x+y}{3}\right) = \frac{4 - 2f(x) - 2f(y)}{3} \qquad \forall, x, y \in R$ ......(1)

differentiate w.r.t. y $\frac{1}{3} f'\left(\frac{x+y}{3}\right) = -\frac{2}{3} f'(y)$

replace x with 3x and y with 0

$f'(x) = -2f'(0)$

put x = 0

we get $f'(0) = 0 \Rightarrow f'(x) = 0 \quad \Rightarrow \quad f(x) = \text{constant} \Rightarrow f(x) = f(0)$

in equation (1) put x = 0 = y it gives $f(0) = \frac{4}{7} \Rightarrow f(x) = \frac{4}{7}$

PART-II

1. $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \frac{\sin^{-1}(1-\{x\}) \ . \ \cos^{-1}(1-\{x\})}{\sqrt{2\{x\}} \ (1-\{x\})} = \lim\limits_{h \to 0} \frac{\sin^{-1}(1-h) \ . \ \cos^{-1}(1-h)}{\sqrt{2h} \ (1-h)}$

$= \lim\limits_{h \to 0} \frac{\sin^{-1}(1-h) \ \sin^{-1}\sqrt{h \ (2-h)}}{(1-h) \ \sqrt{2h}} = \lim\limits_{h \to 0} \frac{\frac{\pi}{2}}{1} \ \frac{\sin^{-1}\sqrt{h(2-h)}}{\sqrt{2h-h^2} \ \sqrt{2h}} \sqrt{2h-h^2} = \lim\limits_{h \to 0} \frac{\pi}{2} \ .1. \ \sqrt{1-\frac{h}{2}} = \frac{\pi}{2}$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \frac{\sin^{-1}(1-\{x\}) \ \cos^{-1}(1-\{x\})}{\sqrt{2 \ \{x\}} \ (1-\{x\})} = \lim\limits_{h \to 0} \frac{\sin^{-1}h \ . \ \cos^{-1}h}{\sqrt{2(1-h)} \ h} = \frac{\frac{\pi}{2}}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$

then $4\left(\lim\limits_{x \to 0^+} f(x)\right)^2 + 8\left(\lim\limits_{x \to 0^-} f(x)\right)^2 = 4(\pi/2)^2 + 8(\pi/2\sqrt{2})^2 = 2\pi^2$

2. $\lim\limits_{x\to\infty}\left(x\sin\left(\frac{1}{x}\right)+\sin\left(\frac{1}{x^2}\right)\right)$ $\lim\limits_{x\to\infty}\left(\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}+\sin\left(\frac{1}{x^2}\right)\right) = 1 + 0 = 1$

3. $\lim\limits_{x\to0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2} = \lim\limits_{x\to0}\frac{1-\cos^2 x\cos 2x}{(1+\cos x\sqrt{\cos 2x})x^2} = \lim\limits_{x\to0}\frac{1}{(1+\cos x\sqrt{\cos 2x})}$.

$\lim\limits_{x\to0}\frac{1-\cos^2 x\cos 2}{x^2} = \frac{1}{2}\lim\limits_{x\to0}\frac{1-\left(\frac{1+\cos 2x}{2}\right)\cos 2x}{x^2} = \frac{1}{2}\lim\limits_{x\to0}\frac{2-\cos 2x-(\cos 2x)^2}{2x^2}$

$= \frac{1}{4}\lim\limits_{x\to0} -\frac{1}{x^2}(\cos 2x + 2)(\cos 2x - 1) = \frac{1}{4}\lim\limits_{x\to0}(\cos 2x + 2).\ \lim\limits_{x\to0}\frac{1-\cos 2x}{x^2}$

$= \frac{(1+2)}{4}.\ \lim\limits_{x\to0}\frac{2\sin^2 x}{x^2} = \frac{3}{4}.2\lim\limits_{x\to0}\left(\frac{\sin x}{x}\right)^2 = \frac{3}{2} \Rightarrow$ limit $= \frac{3}{2}+\frac{30}{2} = 16.50$

4. $\lim\limits_{x\to\infty} f(x)$ exist and is finite & non zero $\lim\limits_{x\to\infty}\left[f(x)+\frac{3f(x)-1}{f^2(x)}\right] = 3 \Rightarrow \lim\limits_{x\to\infty} f(x) + \frac{3\lim\limits_{x\to\infty}f(x)-1}{\left[\lim\limits_{x\to\infty}f(x)\right]^2} = 3$

Let $\lim\limits_{x\to\infty} f(x) = A$

$A + \frac{3A-1}{A^2} = 3 \Rightarrow A = 1$ so $\lim\limits_{x\to\infty} f(x) = 1$

5. $\lim\limits_{x\to0} f(g(h(x)))$

L.H.L. $x \to 0^-$

$\lim\limits_{x\to0^-} h(x) = 0^+$

$\lim\limits_{x\to0^+} f(g(x))$

then $\lim\limits_{x\to0^+} g(x) = 1^+$

$\lim\limits_{x\to0^+} f(x) = 12(1) - \frac{1}{3} = \frac{35}{3}$

R.H.L. $x \to 0^+$

$h(x) = 0^+$

so $\lim\limits_{x\to0^+} f(g(x)) = 0$

L.H.L. = R.H.L. $= \frac{35}{3}$

6. $\lim\limits_{x\to0} g(f(x))$

L.H.L. $= \lim\limits_{x\to0^-} g[f(x)]$

$\lim\limits_{x\to0^-} g(\sin x) = \lim\limits_{h\to0} g(\sin h) = \lim\limits_{h\to0}\left(\sin^2 h + \frac{5}{4}\right) = \frac{5}{4}$

R.H.L. $= \lim\limits_{x\to0^+} g[f(x)] = \lim\limits_{x\to0^+} g(\sin x) = \lim\limits_{h\to0} g(\sin h) = \lim\limits_{h\to0}\left(\sin^2 h + \frac{5}{4}\right) = \frac{5}{4}$

L.H.L. = R.H.L. $= \frac{5}{4}$

so $\lim\limits_{x\to0} g[f(x)] = \frac{5}{4}$

7. $\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2}}+\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+............+\dfrac{1}{\sqrt{n^2+2n}}\right)$

using sandwich theorem

$\dfrac{1}{\sqrt{n^2}} \le \dfrac{1}{n}$

$\dfrac{1}{\sqrt{n^2+1}} \le \dfrac{1}{n}$

$\vdots \qquad \vdots$

$\dfrac{1}{\sqrt{n^2+2n}} \le \dfrac{1}{n}$

adding all these inequilities

$\dfrac{1}{\sqrt{n^2}}+\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+............+\dfrac{1}{\sqrt{n^2+2n}} \le \dfrac{2n}{n}$

Taking both side $\lim\limits_{n\to\infty}$

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+............+\frac{1}{\sqrt{n^2+2n}}\right)=2$$

8. $\lim\limits_{x\to0^+} x^2\left[\dfrac{1}{x^2}\right]=\lim\limits_{x\to0^+} x^2\left(\dfrac{1}{x^2}-\left\{\dfrac{1}{x^2}\right\}\right) \Rightarrow \lim\limits_{x\to0^+}\left(1-x^2\left\{\dfrac{1}{x^2}\right\}\right)=1$

similarly $\lim\limits_{x\to0^-} f(x)=1$

9. $\lim\limits_{x\to0}\left(\dfrac{\sin^{-1}x-\tan^{-1}x}{x^3}+\dfrac{94x\tan^{-1}(\sqrt{2}-1)}{\sin\pi x}\right)$ is equal to

$$\lim_{x\to0}\left(\frac{\sin^{-1}x-\tan^{-1}x}{x^3}+\frac{94x\frac{\pi}{8}}{\sin\pi x}\right)=\lim_{x\to0}\frac{\left[x+\frac{1^2}{3\,!}x^3+....\right]-\left[x-\frac{x^3}{3}+\frac{x^5}{5}-....\right]}{x^3}+\frac{94}{8}=\lim_{x\to0}\frac{x^3\left(\frac{1}{3!}+\frac{1}{3}\right)+x^5\left(\;\right)+......}{x^3}+\frac{47}{4}=12.25$$

10. $\lim\limits_{x\to0}\dfrac{x^3}{\sqrt{a+x}\,(bx-\sin x)}=1 \qquad \Rightarrow \qquad \lim\limits_{x\to0}\dfrac{1}{\sqrt{a+x}}\lim\limits_{x\to0}\dfrac{x^3}{bx-\sin x}=1$

$\Rightarrow \quad \dfrac{1}{\sqrt{a}}.\lim\limits_{x\to0}\dfrac{x^3}{bx-\left[x-\dfrac{x^3}{3\,!}+\dfrac{x^5}{5\,!}-......\right]}=1 \Rightarrow \quad \dfrac{1}{\sqrt{a}}.\lim\limits_{x\to0}\dfrac{x^3}{(b-1)\;x\;+\;\dfrac{x^3}{3\,!}-\dfrac{x^5}{5\,!}+...}=1$

If limit exists, then b – 1= 0 $\qquad \Rightarrow \qquad b=1$

so $\quad \dfrac{1}{\sqrt{a}}.\lim\limits_{x\to0}\dfrac{x^3}{x^3\left[\dfrac{1}{6}-\dfrac{x^2}{120}+....\right]}=1 \qquad \Rightarrow \qquad \dfrac{1}{\sqrt{a}}\times\dfrac{1}{\frac{1}{6}}=1 \quad \Rightarrow \quad a=36$

so $\quad a = 36, \qquad b = 1$

11. $f(x)=\sum\limits_{\lambda=1}^{n}\left(x-\dfrac{7}{\lambda}\right)\left(x-\dfrac{5}{4(\lambda+1)}\right)$ $f(0)=\sum\limits_{\lambda=1}^{n}\left(-\dfrac{7}{\lambda}\right)\left(-\dfrac{5}{4(\lambda+1)}\right)$

$\Rightarrow \quad f(0)=\sum\limits_{\lambda=1}^{n}\left(\dfrac{35}{4(\lambda)(\lambda+1)}\right) \Rightarrow \quad f(0)=\dfrac{35}{4}\sum\limits_{\lambda=1}^{n}\left(\dfrac{1}{\lambda}-\dfrac{1}{\lambda+1}\right)$

$\Rightarrow \quad f(0)=\dfrac{35}{4}\left(1-\dfrac{1}{n+1}\right) \Rightarrow \quad f(0)=\dfrac{35n}{4(n+1)}$

Now $\lim\limits_{n\to\infty} f(0)=\lim\limits_{n\to\infty}\dfrac{35n}{4(n+1)}=\lim\limits_{n\to\infty}\dfrac{35}{4\left(1+\dfrac{1}{n}\right)}=\dfrac{35}{4(1+0)}=\dfrac{35}{4}$

12. LHL = $\lim\limits_{h\to 0^+} f(0-h) \lim\limits_{h\to 0^+} = f(-h) = \lim\limits_{h\to 0^+} (-1)^{[h^2]} = 1$

RHL = $\lim\limits_{h\to 0^+} f(0+h) = \lim\limits_{h\to 0^+} f(h) = \lim\limits_{h\to 0^+}\left(\lim\limits_{n\to\infty}\dfrac{1}{4+h^n}\right) = \dfrac{1}{4}$

13. $\because \quad (1+x)^{\frac{1}{x}} = e\left[1-\dfrac{x}{2}+\dfrac{11}{24}x^2-.....\right]$

Now $\ell = \lim\limits_{x\to 0}\dfrac{e-(1+x)^{\frac{1}{x}}}{\tan x} = \lim\limits_{x\to 0}\dfrac{e-e\left[1-\dfrac{x}{2}+\dfrac{11}{24}x^2-.....\right]}{x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+.....} = \dfrac{e}{2}$

14. $\lim\limits_{x\to 0}\dfrac{e^{-nx}+e^{nx}-2\cos\dfrac{nx}{2}-kx^2}{(\sin x-\tan x)}$

$= \lim\limits_{x\to 0}\dfrac{2\left[1+\dfrac{n^2x^2}{2!}+\dfrac{n^4x^4}{4!}+.....\right]-2\left[1-\dfrac{n^2x^2}{4.2!}+\dfrac{n^4x^4}{16.4!}-........\right]-kx^2}{\left(x-\dfrac{x^3}{3!}+........\right)-\left(x+\dfrac{x^3}{3}+\dfrac{2}{15}x^5+.......\right)}$

$= \lim\limits_{x\to 0}\dfrac{x^2\left(n^2+\dfrac{n^2}{4}-k\right)+x^4\left(\dfrac{2n^4}{4!}-\dfrac{2n^4}{16.4!}\right)}{x^3\left(-\dfrac{1}{3!}-\dfrac{1}{3}\right)+.........}$

limit exists, if coff. of x^2 is zero.

$\Rightarrow n^2+\dfrac{n^2}{4}-k=0 \Rightarrow 4k = 5n^2$

so the possible value match that is n = 2, k = 5

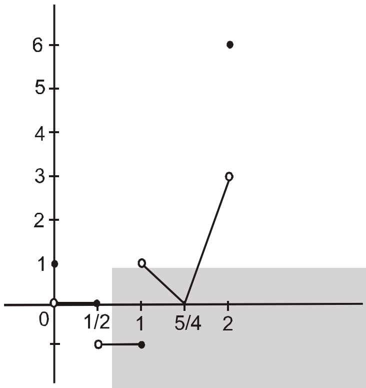
15. $\lim\limits_{n\to\infty}\dfrac{\sum\limits_{r=1}^{n} r^2(n-r+1)}{\sum\limits_{r=1}^{n} r^3} = \lim\limits_{n\to\infty}\dfrac{\sum\limits_{r=1}^{n}(n+1)\ r^2-\sum\limits_{r=1}^{n} r^3}{\sum\limits_{r=1}^{n} r^3} = \lim\limits_{n\to\infty}\left(\dfrac{(n+1)\dfrac{(n)\ (n+1)\ (2n+1)}{6}}{\dfrac{n^2(n+1)^2}{4}}-1\right)$

$= \dfrac{1/3}{1/4}-1 = \dfrac{4}{3}-1 = \dfrac{1}{3}$

16. $\lim\limits_{n\to\infty}\dfrac{n^{98}}{n^{x-1}\left({}^xC_1-\dfrac{{}^xC_2}{2n}+\dfrac{{}^xC_3}{6n^2}-...\right)} = \dfrac{1}{99}$

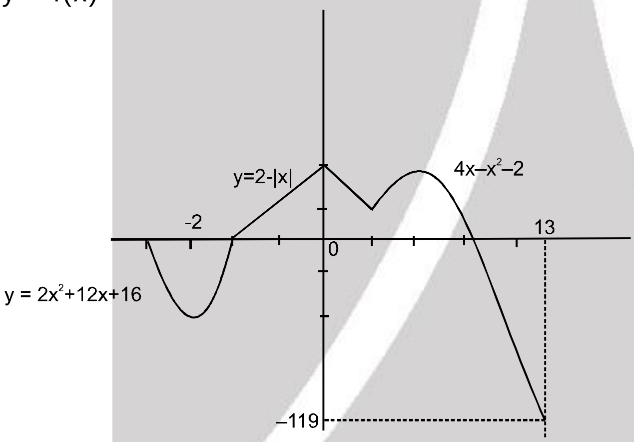
the limit obviously exists if x – 1 = 98

17. $f(x) = \begin{cases} 1 & , \quad x = 0 \\ 0 & , \quad 0 < x \le 1/2 \\ -1 & , \quad 1/2 < x \le 1 \\ 5-4x & , \quad 1 < x < 5/4 \\ 4x-5 & , \quad 5/4 \le x < 2 \\ 6 & , \quad x = 2 \end{cases}$

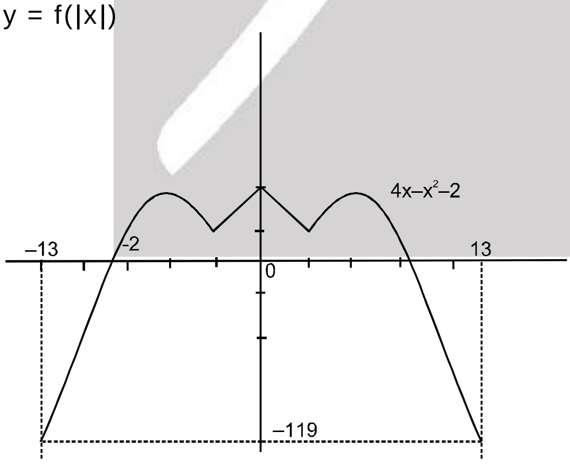


f(x) is discontinous at x = 0, 1/2, 1, 2 in [0, 2]

18. y = f(x)



y = f(|x|)



19. $f\left(\frac{\pi}{2}\right) = \lim_{h\to 0} \frac{1-\cosh}{4h^2} \frac{\ell n\ (\cosh)}{\ell n\ [1+4h^2]}$

$= \lim_{h\to 0} \frac{2}{16\times 16}\left(\frac{\sin^2 h/2}{h^2/2}\right) . \frac{4h^2}{\ell n\ (1+4h^2)} . \frac{\ell n(1-2\sin^2 h/2)}{2\sin^2 h/2} . \frac{2\sin^2 h/2}{h^2/2} = \frac{1}{64} .1.1. (-1).1 = -\frac{1}{64}$

$\Rightarrow \quad \alpha^\beta = 64 \quad = 2^6, 4^3, 8^2, 64^1$

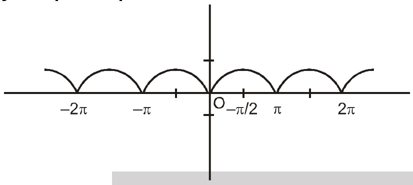
20. We have $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0^+} (\sin(-h) + \cos(-h))^{\text{cosec}(-h)} = \lim\limits_{h\to 0^+} (\cos h - \sin h)^{-\text{cosec}h}$

$$= \lim_{h\to 0^+} (1+(\cos h - \sin h - 1))^{\frac{1}{(\cos h - \sin h - 1)}\cdot\frac{(\cos h - \sin h - 1)}{(-\sin h)}} = \lim_{h\to 0^+} e^{\frac{\cos h - \sin h - 1}{-\sin h}} = e$$

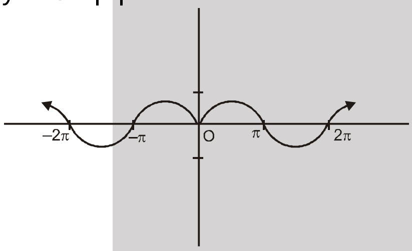
Now we have $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0^+} \dfrac{e^{\frac{1}{h}} + e^{2/h} + e^{3/h}}{ae^{-2+1/h} + be^{-1+3/h}} = \lim\limits_{h\to 0^+} \dfrac{e^{-\frac{2}{h}} + e^{\frac{-1}{h}} + 1}{(ae^{-2})e^{-2/h} + (be^{-1})} = \dfrac{e}{b}$

If 'f' is continuous at x = 0, then $e = a = \dfrac{e}{b}$ gives a = e and b = 1

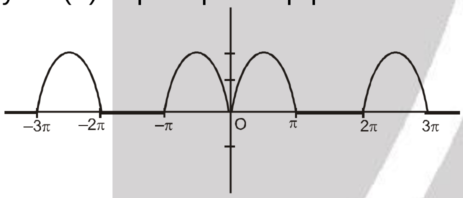
21. $y = |\sin x|$



$y = \sin|x|$



$y = f(x) = |\sin x| + \sin|x|$



f(x) is continous every where
f(x) is not differentiable at x = nπ
f(x) is not periodic

22. Differentiability at x = 1

$$f'(1^-) = \underset{h\to 0}{\text{Lim}} \frac{\left(\frac{\sin[(1-h)^2]}{(1-h)^2 - 3\ (1-h) + 8}\ \pi + a\right)(1-h)^3 + b - (a+b)}{-h}$$

$$= \underset{h\to 0}{\text{Lim}} \frac{a\ (1-h)^3 - a}{-h}\left(\frac{0}{0}\ \text{form}\right) = \underset{h\to 0}{\text{Lim}} \frac{3a\ (1-h)^2}{1}$$

$f'(1^-) = 3a$

$$f'(1^+) = \underset{h\to 0}{\text{Lim}} \frac{2\cos(1+h)\ \pi + \tan^{-1}(1+h) - a - b}{h} = \underset{h\to 0}{\text{Lim}} \frac{(-2\cos\ \pi h + \tan^{-1}(1+h) - a - b)}{h}$$

Function is differentiable

$\therefore \quad -2 + \dfrac{\pi}{4} = a + b \qquad$ .....(1)

$$= \underset{h\to 0}{\text{Lim}} \frac{-2\cos\pi h + \tan^{-1}(1+h) - 2 + \pi/}{h} = \underset{h\to 0}{\text{Lim}}\ 2\pi\ \sin\ \pi h + \frac{1}{1+(1+h)^2} = \frac{1}{2}$$

Now $f'(1^-) = f'(1^+)$ $3a = \dfrac{1}{2}$

$a = \dfrac{1}{6} \qquad$ ....(2)

by (1) and (2) $b = \dfrac{\pi}{4} - \dfrac{13}{6}$

23. $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1} \quad x^2e^{2(x-1)} = 1$

f(1) = 1

$\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1}$ a sgn (x +1) cos2(x–1) + bx² = a.1.1+ b

for continuity a + b = 1

LHD (x = 1) is $\lim\limits_{h\to0} \dfrac{(1-h)^2e^{-2h}-1}{h} = \lim\limits_{h\to0} 2e^{-2h} + he^{-2h} + \left(\dfrac{e^{-2h}-1}{h}\right)$ = 2 + 0 + 2 = 4

RHD (x = 1) is $\lim\limits_{h\to0} \dfrac{a\,sgn(2+h)\cos 2h + b(1+h)^2 - 1}{h} = \lim\limits_{h\to0} \dfrac{a\cos 2h + b + bh^2 + 2bh - (a+b)}{h}$

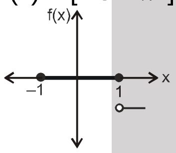
= $\lim\limits_{h\to0} a\left(\dfrac{\cos 2h - 1}{h}\right) + bh + 2b$ = 2b

f(x) is differentiable at x = 1 if 2b = 4 b = 2 a = –1

24. As $0 < \{e^x\} < 1$

$\therefore \quad \lim\limits_{n\to\infty} \dfrac{\{e^x\}^n - 1}{\{e^x\}^n + 1} = -\ 1 \quad \Rightarrow \quad f(x) = -1\ \forall\ x \in R$

25. f(x) = [x sin πx] graph of f(x) is as shown in the figure



26. Given f″(0) = 4 $\lim\limits_{x\to0} \dfrac{2f(x) - 3f(2x) + f(4x)}{x^2} \left[\dfrac{0}{0}\right]$ form

using L′Hospital rule $\lim\limits_{x\to0} \dfrac{2f'(x) - 6f'(2x) + 4f'(4x)}{2x} \left[\dfrac{0}{0}\right]$ form

using L′ Hospital rule $\lim\limits_{x\to0} \dfrac{2f''(x) - 12f''(2x) + 16f''(4x)}{2} = \dfrac{2.4 - 12.4 + 16.4}{2}$ =12

27. f(10 – x) = f(x) = f(4 – x) ⇒ f(10 – x) = f(4 – x)

Let 4 – x = t ⇒ f(6 + t) = t

⇒ f(x) is periodic with period 6. ⇒ f(x) = 101 at x = 0,6,12, 18, 24, 30

Since f(2 + x) = f(2 – x) ⇒ f(x) is symmetric about x = 2

⇒ f(0) = f(4) ⇒ using periodic nature

f(x) = 101 at x = 4, 10, 16, 22, 28 ⇒ f(5 + x) = f(5 – x)

x is symmetric about x = 5

f(0) = f(10) ⇒ x = 4, 10, 16, 22

f(6) = f(4) ⇒ x = 0, 6, 12, 18,

Total different values of x are 0,4, 6, 10, 12, 16, 18, 22, 24, 28, 30

28. $\sum\limits_{k=1}^{n} f(a+k) = 2048\ (2^n - 1)$

or f(a + 1) + f(a + 2) + ... + f(a + n) = 2048 (2^n – 1)

f(x + y) = f(x) . f(y)

f(0) = 1, f(1) = 2

or f(x) = 2^x

Now f(a + 1) + f(a + 2) + ... + f(a + n)

= 2^a [2 + 4 + ... + 2^n] = 2^a . 2(2^n – 1)

or 2048 = 2^{a+1} or a = 10

PART - III

1. $f(x) = \dfrac{x^2 - 9x + 20}{x - [x]}$

$$\lim_{x \to 5^-} \frac{x^2 - 9x + 20}{x - [x]} = \frac{25 - 45 + 20}{1} = 0$$

$$\lim_{x \to 5^+} \frac{x^2 - 9x + 20}{x - [x]} = \lim_{h\to0} \frac{(5+h)^2 - 9(5+h) + 20}{5 + h - [5+h]} = \lim_{h\to0} \frac{25 + 10h + h^2 - 45 - 9h + 20}{h}$$

$$= \lim_{h\to0} \frac{h^2 + h}{h} = \lim_{h\to0} \frac{h(h+1)}{h} = 1$$

$\because$ $\lim_{x \to 5^-} f(x) \neq \lim_{x \to 5^+} f(x)$

so $\lim_{x \to 5} f(x)$ does not exist

2. Let $f(x) = \dfrac{\cos 2 - \cos 2x}{x^2 - |x|}$

(A) $\lim_{x\to-1} f(x)$

for $x = -1$ $\qquad |x| = -x$

$f(x) = \dfrac{\cos 2 - \cos 2x}{x^2 + x}$

Now $\lim_{x\to-1} \dfrac{\cos 2 - \cos 2x}{x^2 + x}$ (form) $= \lim_{x\to-1} \dfrac{2\sin 2x}{2x+1} = 2\sin 2$

(B) $\lim_{x\to1} \dfrac{\cos 2 - \cos 2x}{x^2 - x}$ ($\dfrac{0}{0}$ form) $= \lim_{x\to1} \dfrac{2\sin 2x}{2x - 1} = 2\sin 2$

3. For $\lim_{x \to 0} \dfrac{1 + a\cos x}{x^2}$ for $\left(\dfrac{0}{0}\right)$ form

$1 + a = 0 \quad \Rightarrow \quad a = -1$

for $\lim_{x \to 0} \dfrac{b\sin x}{x^3}$ $\dfrac{b}{x^2} = \lim_{x \to 0} b = 0$

Now, $\ell = \lim_{x \to 0} \dfrac{1 + a\cos x}{x^2} - \lim_{x\to0} \dfrac{b\sin x}{x^3} = \lim_{x \to 0} \dfrac{1 - \cos x}{x^2} = \lim_{x \to 0} \dfrac{2\sin^2 \frac{x}{2}}{x^2} = \lim_{x \to 0} \dfrac{2}{4}\left(\dfrac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = \dfrac{1}{2} \cdot (1)^2 = \dfrac{1}{2}$

$\therefore$ $(a, b) = (-1, 0)$ and $\ell = \dfrac{1}{2}$

4. $f(x) = \dfrac{|x + \pi|}{\sin x}$

(A) $f(-\pi^+) = \lim_{h\to0} \dfrac{|-\pi + h + \pi|}{\sin(-\pi + h)} = \lim_{h\to0} \dfrac{|h|}{-\sin h} = -1$

(B) $f(-\pi^-) = \lim_{h\to0} \dfrac{|-\pi - h + \pi|}{\sin(-\pi - h)} = \lim_{h\to0} \dfrac{|h|}{\sin h} = 1$

(C) $f(-\pi^+) \neq f(-\pi^-)$ $\qquad$ so $\lim_{x \to -\pi} f(x)$ does not exist

(D) for $\lim_{x \to \pi} f(x)$

LHL $= \lim_{x\to\pi^-} \dfrac{|x + \pi|}{\sin x} = \lim_{h\to0} \dfrac{2\pi - h}{\sin h} = \dfrac{2\pi}{0} = \infty$

RHL $= \lim_{x\to\pi^+} \dfrac{|x + \pi|}{\sin x} = \lim_{h\to0} \dfrac{2\pi + h}{-\sin h} = -\dfrac{2\pi}{0} = -\infty$

LHL $\neq$ RHL $\quad$ so $\quad \lim_{x\to\pi} f(x)$ does not exist.

5. $f(x) = \begin{cases} 1+\dfrac{2x}{a}, & 0 \le x < 1 \\ ax, & 1 \le x < 2 \end{cases}$

L.H.L. $= \lim\limits_{x\to 1^-} f(x) = \lim\limits_{h\to 0} 1 + \dfrac{2(1-h)}{a} = 1 + \dfrac{2}{a}$

R.H.L. $= \lim\limits_{x\to 1^+} f(x) = \lim\limits_{h\to 0} a(1+h) = a$

$\because$ f(x) exists

L.H.L.= R.H.L.

$\Rightarrow$ $1 + \dfrac{2}{a} = a \Rightarrow a^2 - a - 2 = 0 \Rightarrow (a-2)(a+1) = 0 \Rightarrow a = 2, -1$

6. α, β be the roots of the equation $ax^2 + bx + c = 0$

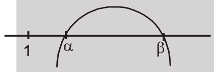
where $1 < \alpha < \beta$.

(i) if $a > 0$

So $ax^2 + bx + c > 0$ when $x \in (-\infty, \alpha) \cup (\beta, \infty)$

So $\lim\limits_{x\to x_0} \dfrac{|ax^2+bx+c|}{ax^2+bx+c} = 1$ when $a > 0$ and $x \in (-\infty, \alpha) \cup (\beta, \infty)$

(ii) If $a < 0$



So $ax^2 + bx + c > 0$ when $x \in (\alpha, \beta)$ $\lim\limits_{x\to x_0} \dfrac{|ax^2+bx+c|}{ax^2+bx+c} = 1$

when $a < 0$ and $x \in (\alpha, \beta)$

So (A) $a > 0$ and $x_0 < 1$ right

(B) $a > 0$ and $x_0 > \beta$ right

(C) $a < 0$ and $\alpha < x_0 < \beta$ right

(D) $a < 0$ and $x_0 < 1$ wrong

7. (A) If $m > n$ $\Rightarrow m - n > 0$ $\therefore$ $\lim\limits_{x\to 0} \phi(x) = 0$

(B) If m = n, then

$$\lim_{x\to 0} \phi(x) = \lim_{x\to 0} \frac{a_0x^m + a_1^{m+1} + \dots\dots\dots + a_kx^{m+k}}{b_0x^m + b_1^{m+1} + \dots\dots\dots + b_\ell x^{m+\ell}} = \frac{a_0}{b_0}$$

(C) If $n - m$ is even positive, then $\lim\limits_{x\to 0} \phi(x) = \infty$ as $\dfrac{a_0}{b_0} > 0$

(D) If $n - m$ is even positive and $\dfrac{a_0}{b_0} < 0$, then $\lim\limits_{x\to 0} \phi(x) = -\infty$

8. To find $\lim\limits_{x\to 0} f(x)$

L.H.L. $= \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \sqrt{\{x\} \cot \{x\}} = \lim\limits_{h\to 0} \sqrt{(1-h) \cot (1-h} = \sqrt{\cot 1}$

R.H.L. $= \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{\tan^2[x]}{x^2 - [x]^2} = \lim\limits_{h\to 0} \dfrac{\tan^2[0+h]}{(0+h)^2 - [0+h]^2} = \lim\limits_{h\to 0} \dfrac{\tan^2 0}{h^2} = 0$

$\because$ L.H.L. $\neq$ R.H.L.so f(x) does not exist. f(x) is not continuous at x = 0.

Now $\cot^{-1} \left(\lim\limits_{x\to 0^-} f(x)\right)^2 = \cot^{-1} (\sqrt{\cot 1})^2 = \cot^{-1} (\cot 1) = 1$

9. (A) $\lim\limits_{x \to -\infty} \dfrac{\sqrt{x^2+2}}{3x-6} = \lim\limits_{x \to \infty} \dfrac{\sqrt{x^2+2}}{-3x-6} = \lim\limits_{x \to \infty} \dfrac{\sqrt{1+\frac{2}{x^2}}}{-3-\frac{6}{x}} = -\dfrac{1}{3}$

(B) $\lim\limits_{x \to \infty} \dfrac{\sqrt{x^2+2}}{3x-6} = \lim\limits_{x \to \infty} \dfrac{\sqrt{1+\frac{2}{x^2}}}{3-\frac{6}{x}} = \dfrac{1}{3}$

10. $\lim\limits_{x \to 0} \dfrac{2\sin x \cos x + a \sin x}{x^3} = p$ (finite)

$\Rightarrow \lim\limits_{x \to 0} \dfrac{\sin x}{x} . \left(\dfrac{2\cos x + a}{x^2}\right) = p \Rightarrow \lim\limits_{x \to 0} \dfrac{\sin x}{x} . \lim\limits_{x \to 0} \dfrac{2\cos x + a}{x^2} = p \Rightarrow \lim\limits_{x \to 0} \dfrac{2\cos x + a}{x^2} = p$

For $\left(\dfrac{0}{0}\right)$ form $2 + a = 0$

$a = -2$

$\Rightarrow \quad p = \lim\limits_{x \to 0} \dfrac{2(\cos x - 1)}{x^2} \quad \Rightarrow \quad p = \lim\limits_{x \to 0} \dfrac{-2\left(2\sin^2 \frac{x}{2}\right)}{x^2} \Rightarrow \quad p = -1$

11. $\lim\limits_{x \to \infty} \dfrac{(ax+1)^n}{x^n + A}$

(A) If $n \in N \Rightarrow \lim\limits_{x\to\infty} \dfrac{\left(a+\frac{1}{x}\right)^n}{1+\frac{A}{x^n}} = \dfrac{(a+0)^n}{1+0} = a^n$

(B) If $n \in Z^-$ & $a = A = 0$ then $\lim\limits_{x\to\infty} \dfrac{(ax+1)^n}{x^n + A} = \lim\limits_{x\to\infty} \dfrac{1}{x^n} = \infty$ $n \in Z^-$

(C) If $n = 0$ then $\lim\limits_{x\to\infty} \dfrac{(ax+1)^n}{x^n + A} = \lim\limits_{x\to\infty} \dfrac{1}{1+A} = \dfrac{1}{1+A}$

(D) If $n \in Z^-$, $A = 0$ & $a \neq 0$ then $\lim\limits_{x\to\infty} \dfrac{(ax+1)^n}{x^n + A} = \lim\limits_{x\to\infty} \dfrac{(ax+1)^n}{x^n} = \lim\limits_{x\to\infty} \left(a+\dfrac{1}{x}\right)^n$

$= (a + 0)^n = a^n$

12. **(i)** $\ell = \lim\limits_{x\to\infty} \left(\sin\sqrt{x+1} - \sin\sqrt{x}\right)$

$\Rightarrow \quad \ell = \lim\limits_{x\to\infty} 2\cos\left(\dfrac{\sqrt{x+1}+\sqrt{x}}{2}\right) . \sin\left(\dfrac{\sqrt{x+1}-\sqrt{x}}{2}\right)$

$\Rightarrow \quad \ell = \lim\limits_{x\to\infty} 2\cos\left(\dfrac{\sqrt{x+1}+\sqrt{x}}{2}\right) . \sin\left(\dfrac{x+1-x}{2\left(\sqrt{x+1}+\sqrt{x}\right)}\right)$

$\Rightarrow \quad \ell = \lim\limits_{x\to\infty} 2\cos\left(\dfrac{\sqrt{x}+\sqrt{x+1}}{2}\right) \sin\left(\dfrac{1}{2\left(\sqrt{x}+\sqrt{x+1}\right)}\right)$

$\Rightarrow \quad \ell$ = (oscillating value –1 to 1) $\times 0 = 0$

(ii) $m = \lim\limits_{x\to-\infty} \left[\sin\sqrt{x+1} - \sin\sqrt{x}\right]$

when x

then $\sqrt{x}$ undefined

m is undefined

13. $\lim_{x \to 0^-} f(x) = \lim_{h\to 0} |0-h|^{\sin(0-h)} = \lim_{h\to 0} h^{\sin(-h)} = e^{\lim_{h\to 0}(-\sin h)(\ell n h)} = e^0 = 1$

$\lim_{x \to 0^+} f(x) = \lim_{h\to \infty} |0+h|^{\sin(0+h)} = \lim_{h\to\infty} h^{\sin h} = e^{\lim_{h\to\infty}(\sin h)(\ell n h)} = e^0 = 1$

$\because \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 1$

$\therefore \quad \lim_{x \to 0} f(x) = 1$

14. $\lim_{x \to 0} (\cos x + a \sin bx)^{\frac{1}{x}} = e^2 \qquad (1^\infty \text{ form})$

$\Rightarrow \quad e^{\lim_{x\to 0}\left(\frac{\cos x + a\sin bx - 1}{x}\right)} = e^2 \Rightarrow e^{\lim_{x\to 0}\left(\frac{-\sin x + ab\cos bx}{1}\right)} = e^2 \Rightarrow \lim_{x\to 0} (-\sin x + ab\cos bx) = 2$

$\Rightarrow \quad ab = 2$

$\therefore \quad a = 1, b = 2; \quad a = 2, b = 1; \quad a = 3, b = \frac{2}{3}; \ a = \frac{2}{3}, b = 3$

15. $\lim_{x \to 0} (1 + ax + bx^2)^{\frac{2}{x}} = e^3 \qquad (1^\infty \text{ form})$

$\Rightarrow \quad e^{\lim_{x\to 0} (ax+bx^2)\frac{2}{x}} = e^3 \Rightarrow \lim_{x\to 0} \frac{2x(a + bx)}{x} = 3$

$\Rightarrow \quad \lim_{x\to 0} 2(a+bx) = 3 \quad \Rightarrow \quad 2a = 3$

$\Rightarrow \quad a = \frac{3}{2}, \ b \in R$

16. $\lim_{x\to 0^+} \log_{\sin\frac{x}{2}}(\sin x) = \lim_{h\to 0} \log_{\sin\left(\frac{0+h}{2}\right)}(\sin(0+h)) = \lim_{h\to 0} \log_{\sin\left(\frac{h}{2}\right)}(\sin h) = \lim_{h\to 0} \frac{\log_e \sin h}{\log_e \sin\frac{h}{2}}$

$= \lim_{h\to 0} \frac{\log_e 2\sin\frac{h}{2}\cos\frac{h}{2}}{\log_e \sin\frac{h}{2}} = \lim_{h\to 0} \frac{\log_e 2 + \log_e \sin\frac{h}{2} + \log_e \cos\frac{h}{2}}{\log_e \sin\frac{h}{2}} = \lim_{h\to 0} 1 + \frac{\log_e 2 + \log_e \cos\frac{h}{2}}{\log_e \sin\frac{h}{2}}$

$= 1 + \frac{\log_e 2 + 0}{\infty} = 1 + 0 = 1$

17. $\lim_{x\to\infty} \frac{x^n}{e^x} = 0 \qquad (n \in \text{integer})$

case (i) when $n = 0$

then $\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$

case(ii) when n is +ve integer $\lim_{x\to\infty} \frac{x^n}{e^x} \left(\frac{\infty}{\infty} \text{ form}\right) = \lim_{x\to\infty} \frac{n!}{e^x} = 0$

case(iii) when n is – ve $n = -m$ where $m \in z^+$

$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{x^{-m}}{e^x} = \lim_{x\to\infty} \frac{1}{x^m . e^x} = 0$

so $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$

18. $f(1) = \lim_{n\to\infty} \frac{\log 3 - 1^{2n}\sin 1}{1^{2n}+1} = \frac{\log 3 - \sin 1}{2}$

$f(1^+) = \lim_{h\to 0} \lim_{n\to\infty} \frac{\log(3+h) - (1+h)^{2n} - \sin(1+h)}{(1+h)^{2n}+1} = -\sin 1$

$f(1^-) = \lim_{h\to 0} \lim_{n\to\infty} \frac{\log(3+h) - (1-h)^{2n} - \sin(1+h)}{(1-h)^{2n}+1} = \log 3$

discontinous at x = 1

19. (A) f(x) is continuous no where

(B) g(x) is continuous at x = 1/2

(C) h(x) is continuous at x = 0

(D) k(x) is continuous at x = 0

20. f(x) = $\begin{bmatrix} |x-3| & x \ge 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4} & x < 1 \end{bmatrix}$ Clearly it is continuous at x = 1

$f(1^+) = f(1^-) = f(1)$ at x = 3 $f(3^+) = f(3^+) = f(3) = 0$

It is continuous at x = 3

$f(1^+) = \lim\limits_{n \to 0} \frac{f(1+h) - f(1)}{-h} = -1$

$f(1^-) = \lim\limits_{n \to 0} \frac{f(1-h) - f(1)}{-h} = \frac{h^2 - 2h + 1 - 6 + 6h + 13 - 8}{-4h} = \lim\limits_{n \to 0} \frac{h^2 + 4h}{-4h} = \Rightarrow -1$

21. $f(x) = \frac{1}{2}\, x - 1 \qquad \forall \quad x \in [0, \pi]$

$\frac{1}{f(x)} = \frac{2}{x-2}$ which is not continous at x = 2

$\tan f(x) = \tan\left(\frac{x-2}{2}\right) \qquad \because \quad x \in [0, \pi]$

$\therefore \quad \frac{x-2}{2} \in \left[-1, \frac{\pi - 2}{2}\right]$

$\Rightarrow$ tanf(x) is continuous in [0, π] y = f^{-1}(x) = 2(1+x), which is also continous in [0, π]

22. f(x) = [x] and $g(x) = \begin{cases} 0\,, & x \in I \\ x^2\,, & x \in R - I \end{cases}$

$\lim\limits_{x \to 1} g(x) = \lim\limits_{x \to 1} x^2 = 1$, but g(1) = 0

$\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} [x]$ does not exist since LHL = 0 and RHL = 1

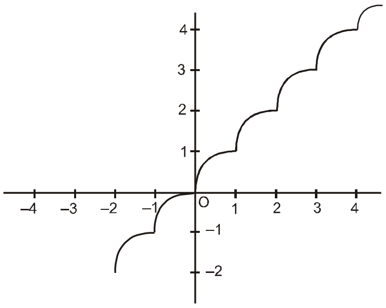
gof(x) = g([x]) = 0 $\Rightarrow$ gof(x) is continous for all values of x

fog = $\begin{cases} 0 & , \quad x \in I \\ [x^2] & , \quad x \in R - I \end{cases}$

fog(1) = 0 , $\lim\limits_{x \to 1^-} \text{fog}(x) = 0$, $\lim\limits_{x \to 1^+} \text{fog}(x) = 1$

fog is not continous at x = 1

23. f(x) = [x] + $\sqrt{\{x\}}$ Curve of y = f(x) = $\begin{cases} -1+\sqrt{x+1} & , -1\le x<0 \\ \sqrt{x} & , 0\le x<1 \\ 1+\sqrt{x-1} & , 1\le x<2 \end{cases}$



Method: II y = f(x) can be discontinuous only at x $\in$ I

so we check continuity only at x = n $\in$ I

f(n) = [n] + $\sqrt{\{n\}} = n + 0 = n$

LHL (x = n) is $\lim\limits_{h\to 0}$ [n – h] + $\sqrt{\{n-h\}}$ = (n–1) + 1 = n

RHL (x = n) is $\lim\limits_{h\to 0}$ [n + h] + $\sqrt{\{n+h\}}$ = (n+0) = n

f(x) is continous for x $\in$ R

24. f(x) = $\left|x-\frac{1}{2}\right|$ + |x–1| + tanx

$\left|x-\frac{1}{2}\right|$ is non-differentiable at x = $\frac{1}{2}$

|x–1| is non-differentiable at x = 1

tan x is non-differentiable at x = $\frac{\pi}{2}$

25. $y = f\ \ (x) = \begin{cases} (\sin^{-1}x)^2 \ \ \cos 1/x & x\ne 0 \\ 0 & x=0 \end{cases}$

f(x) can be discontinuous only at x = 0 in [–1,1]

So we check only at x = 0

LHD (x = 0) = $\lim\limits_{h\to 0} h \ \ \dfrac{(\sin^{-1}(-h))^2 \cos\left(-\frac{1}{h}\right) - 0}{-h}$

$\lim\limits_{h\to 0}\left(\dfrac{\sin^{-1}(-h)}{h}\right)^2 . \ \ h\cos\left(\dfrac{1}{h}\right)$ = –1. 0. [finite quantity between [–1, 1]] = 0

RHD (x = 0) is $\lim\limits_{h\to 0} h \ \ \dfrac{(\sin^{-1}(-h))^2}{h^2} \ . \ \ \cos\left(\dfrac{1}{h}\right) = 0$

Hence f(x) is differentiable as well as continuous in [–1,1]

26. f(x) = $\sum\limits_{k=0}^{n} a_k |x|^k$ = $a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3 + ... + a_n|x|^n$

f(0) = a_0 we know that $\lim\limits_{x\to 0}|x| = 0$

$\lim\limits_{x\to 0} f(x) = a_0$

f(x) is continous for x = 0

$|x|^n$ is differentiable if $n \ne 1$, $n \in N$

f(x) is not differentiable at x = 0, due to presence of |x|

If all a_{2k+1} = 0, f(x) does not contains |x| $\Rightarrow$ f(x) is differentiable at x = 0

27. $f(xy + 1) = f(yx + 1)$
$f(x)\, f(y) - f(y) - x + 2 = f(y)\, f(x) - f(x) - y + 2$
$f(x) - f(y) = x - y$
Putting $y = 0$
$f(x) - 1 = x - 0$
$f(x) = x + 1$

28. Given $f(x+y) = f(x) + f(y) + xy$ ....(1)
and $\lim\limits_{h\to 0} \dfrac{f(h)}{h} = 3$

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(h)+xh}{h} = 3 + x \Rightarrow f(x) = 3x + \dfrac{x^2}{2} + c$

in equation (1) put $x=0=y \Rightarrow f(0) = 0$

Therefor $f(x) = 3x + \dfrac{x^2}{2}$

29. $f(x + a) = \dfrac{1}{2} + \sqrt{f(x) - f((x))^2}$ or $f^2(x + a) + \dfrac{1}{4} - f(x + a) = f(x) - f^2(x)$

or $f^2(x) + f^2(x + a) + \dfrac{1}{4} = f(x) + f(x + a)$ ......(i)

Put $x \to x + a$

$f^2(x + a) + f^2(x + 2a) + \dfrac{1}{4} = f(x + a) + f(x + 2a)$ ....(ii)

from (i) – (ii)
$f^2(x) - f^2(x + 2a) = f(x) - f(x + 2a)$ or $(f(x) - f(x + 2a))\,(f(x) + f(x + 2a) - 1) = 0$
or $f(x) = f(x + 2a)$ $\Rightarrow$ period = 2a

PART–IV

(1 to 3)

$\lim\limits_{x\to 0^+} f(x)$

$= \lim\limits_{x\to 0^+} \dfrac{\sin x + ae^x + be^{-x} + c\,\ell n(1+x)}{x^3} = \lim\limits_{x\to 0^+}$

$$\frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right) + a\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots..\right) + b\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots.\right) + c\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots.\right)}{x^3}$$

$$= \lim_{x\to 0^+} \frac{(a+b) + x(1+a-b) + x^2\left(\frac{a}{2!} + \frac{b}{2!} - \frac{c}{2}\right) + x^3\left(-\frac{1}{3!} + \frac{a}{3!} - \frac{b}{3!} + \frac{c}{3}\right) + x^4\left(\quad\right) + \ldots.}{x^3}$$

$\because$ limit is finite $\Rightarrow \left.\begin{array}{l} a+b=0 \\ 1+a-b=0 \\ a+b-c=0 \end{array}\right\} \Rightarrow a = -\dfrac{1}{2},\ b = \dfrac{1}{2}$ and $c = 0$

Also $\lim\limits_{x\to 0^+} f(x) = -\dfrac{1}{3!} + \dfrac{a}{3!} - \dfrac{b}{3!} + \dfrac{c}{3} = -(1 - a + b - 2c) = -\dfrac{1}{3}$

1. $a + b + c = -\dfrac{1}{2} + \dfrac{1}{2} + 0 = 0$

2. $\lim\limits_{x\to 0^+} f(x) = -\dfrac{1}{3}$

3. $\lim\limits_{x\to 0^+} x\, f(x) = \lim\limits_{x\to 0^+} \dfrac{\sin x + ae^x + be^{-x} + c\ell n(1+x)}{x^2} = \dfrac{a+b-c}{2} = 0$

4. (A) $f(x) = \dfrac{1}{\ell n|x|}$ LHL = 0 = RHL

(B) $f(x) = x\sin\dfrac{\pi}{x}$ LHL = 0 = RHL

(C) $f(x) = \dfrac{1}{1+2^{\cot x}}$ f (0) = not define

LHL = 1

RHL = 0 $\Rightarrow$ LHL $\neq$ RHL

(D) $f(x) = \cos\left(\dfrac{|\sin x|}{x}\right)$ LHL (at x = 0) = cos (– 1) = cos 1

RHL (at x = 0) = cos 1

LHL = RHL

5. (A) $\lim\limits_{x \to 0^-} f(x) = 1$ $\lim\limits_{x \to 0^+} f(x) = 0$

(B) $\lim\limits_{x \to 0^-} f(x) = -\dfrac{\pi}{2}$ $\lim\limits_{x \to 0^+} f(x) = \pi/2$

(C) $\lim\limits_{x \to 0^-} f(x) = -1$ $\lim\limits_{x \to 0^+} f(x) = 1$

(D) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = 0$

6. $f(x)\begin{bmatrix} \tan^{-1}\tan x & x \le \dfrac{\pi}{4} \\ n\ \ [x]+1 & x > \dfrac{\pi}{4} \end{bmatrix}$

$f\left(\dfrac{\pi}{4}^{+}\right) = 1$

$f\left(\dfrac{\pi}{4}^{-}\right) = x = \dfrac{\pi}{4}$

Jump $= 1 - \dfrac{\pi}{4}$

7. $g(t) = \lim\limits_{x \to 0} (1 + a\tan x)^{t/x}$

$g(t) = e^{\lim\limits_{x \to 0} \frac{t}{x} a\tan x} = e^{\lim\limits_{x \to 0} ta\frac{\tan x}{x}}$

$g(t) = e^{ta}$ $= e^{ta}$

$g(x) = e^{ax}$

$\because$ a = 2, $g(x) = e^{2x}$

$g(2) = e^4$

8. $f(x) = \begin{cases} x\ \ e^{ax} & , \ x \le 0 \\ x + ax^2 - x^3 & , \ x > 0 \end{cases}$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x + ax^2 - x^3 = 0$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} x\,e^{ax} = 0$

$f(0) = 0$

$a \in (0, \infty)$

9. $f'(0^+) = \lim\limits_{h\to 0^+} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0^+} \dfrac{h + ah^2 - h^3}{h} = 1$

$f'(0^-) = \lim\limits_{h\to 0^-} \dfrac{f(o-h) - f(o)}{-\ \ h} = \lim\limits_{h\to 0^-} \dfrac{he^{-ah}}{h} = 1$

10. **(10 to 12)**

Given function f(x) can be rewritten as, $f(x) = \begin{cases} 0 & , \quad x < -1 \\ 1+x & , \quad -1 \le x \le 0 \\ 1-x & , \quad 0 < x \le 1 \\ 0 & , \quad x > 1 \end{cases}$

$\Rightarrow \quad f(x-1) = \begin{cases} 0 & , \quad x-1 < -1 \\ 1+(x-1) & , \quad -1 \le x-1 \le 0 \\ 1-(x-1) & , \quad 0 < x-1 \le 1 \\ 0 & , \quad x-1 > 1 \end{cases}$ or $f(x-1) = \begin{cases} 0 & , \quad x < 0 \\ x & , \quad 0 \le x \le 1 \\ 2-x & , \quad 1 < x \le 2 \\ 0 & , \quad x > 2 \end{cases}$

also, $f(x+1) = \begin{cases} 0 & , \quad x+1 < -1 \\ 1+(x+1) & , \quad -1 \le x+1 \le 0 \\ 1-(x+1) & , \quad 0 < x+1 \le 1 \\ 0 & , \quad x+1 > 1 \end{cases}$ or $f(x+1) = \begin{cases} 0 & , \quad x < -2 \\ 2+x & , \quad -2 \le x \le -1 \\ -x & , \quad -1 < x \le 0 \\ 0 & , \quad x > 0 \end{cases}$

Now, $g(x) = f(x-1) + f(x+1) = \begin{cases} 0 & , \quad x < -2 \\ 2+x & , \quad -2 < x \le -1 \\ -x & , \quad -1 < x \le 0 \\ x & , \quad 0 < x \le 1 \\ 2-x & , \quad 1 < x \le 2 \\ 0 & , \quad x > 2 \end{cases}$

It is easy to check that g(x) is continuous for all $x \in R$ and non-differentiable at x = – 2, –1, 0, 1, 2.

EXERCISE # 3

PART - I

1. $\lim\limits_{x\to 0} e^{\frac{x\ell n(1+b^2)}{x}} = 1 + b^2 = 2b \sin^2\theta \Rightarrow \quad \sin^2\theta = \frac{1}{2}\left(b+\frac{1}{b}\right)$

We know $b + \frac{1}{b} \ge 2 \Rightarrow \sin^2\theta \ge 1$ but $\sin^2\theta \le 1 \Rightarrow \sin^2\theta = 1 \Rightarrow \theta = \pm\frac{\pi}{2}$

2. f(x) = kx
Hence f(x) is continuous & differentiable at $x \in \mathbf{R}$ & $f'(x) = k$ (constant)

3*. (A) at $x = -\frac{\pi}{2}$ $\quad Lf\left(-\frac{\pi}{2}\right) = 0 = f\left(-\frac{\pi}{2}\right)$

$Rf\left(-\frac{\pi}{2}\right) = 0 \Rightarrow$ continuous

(B) at x = 0 $\quad Rf'(0) = 1$
$Lf'(0) = 0 \Rightarrow$ not differentiable

(C) at x = 1 $\quad Rf'(1) = 1$
$Lf'(1) = 1 \Rightarrow$ differentiable at x = 1

(D) at $x = -\frac{3}{2} > -\frac{\pi}{2} \Rightarrow f(x) = -\cos x \Rightarrow$ differentiable at $x = -\frac{3}{2}$

4. $f(x) = \frac{x-b}{bx-1} = \frac{1}{b} + \frac{\left(\frac{1}{b}-b\right)}{(bx-1)}$ $\quad f'(x) = \frac{-\left(\frac{1}{b}-b\right)}{(bx-1)^2} b$, $f'(x) < 0 \ \forall\ x\ (0, 1)$

Range of f(x) is (–1, b) so range ≠ co-domain
so f is not invertible
f^{-1} doesnot exist
No comparison with f^{-1}

5. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+x+1}{x+1}-ax-b\right) = 4$

$\lim\limits_{x\to\infty}\left(\dfrac{x^2(1-a)+x(1-a-b)+(1-b)}{x+1}\right) = 4$

Limit is finite. It exists when 1 – a = 0 ⇒ a = 1

then $\lim\limits_{x\to\infty}\left(\dfrac{1-a-b+\dfrac{1-b}{x}}{1+\dfrac{1}{x}}\right) = 4$

$\therefore$ 1 – a – b = 4 ⇒ b = – 4

6. $((1 + a)^{1/3} - 1)x^2 + ((a+1)^{1/2} - 1)x + ((a+1)^{1/6} - 1) = 0$

let $a + 1 = t^6$

$\therefore$ $(t^2 - 1)x^2 + (t^3 - 1)x + (t - 1) = 0$

$(t + 1)x^2 + (t^2 + t + 1)x + 1 = 0$

As $a \to 0$, $t \to 1$

$2x^2 + 3x + 1 = 0 \Rightarrow x = -1$ and $x = -\dfrac{1}{2}$

7. **(i)** for derivability at x = 0

L.H.D. $= f'(0^-) = \lim\limits_{h\to0^+}\dfrac{f(0-h)-f(0)}{-h} = \lim\limits_{h\to0^+}\dfrac{h^2.\left|\cos\left(-\dfrac{\pi}{h}\right)\right|-0}{-h} = \lim\limits_{h\to0^+} -h\,.\left|\cos\dfrac{\pi}{h}\right| = 0$

RHD $f'(0^+) = \lim\limits_{h\to0^+}\dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to0^+}\dfrac{h^2.\left|\cos\left(\dfrac{\pi}{h}\right)\right|-0}{h} = 0 \Rightarrow$ So f(x) is derivable at x = 0

(ii) check for derivability at x = 2

RHD $= f'(2^+) = \lim\limits_{h\to0^+}\dfrac{f(2+h)-f(2)}{h} = \lim\limits_{h\to0^+}\dfrac{(2+h)^2.\left|\cos\left(\dfrac{\pi}{2+h}\right)\right|-0}{h} = \lim\limits_{h\to0^+}\dfrac{(2+h)^2.\cos\left(\dfrac{\pi}{2+h}\right)}{h}$

$= \lim\limits_{h\to0^+}\dfrac{(2+h)^2.\sin\left(\dfrac{\pi}{2}-\dfrac{\pi}{2+h}\right)}{h} = \lim\limits_{h\to0^+}\dfrac{(2+h)^2.\sin\left(\dfrac{\pi h}{2(2+h)}\right)}{\left(\dfrac{\pi}{2(2+h)}\right)h}.\dfrac{\pi}{2(2+h)} = (2)^2\,.\,\dfrac{\pi}{2(2)} = \pi$

LHD $= \lim\limits_{h\to0^+}\dfrac{f(2-h)-f(2)}{-h} = \lim\limits_{h\to0^+}\dfrac{(2-h)^2.\left|\cos\left(\dfrac{\pi}{2-h}\right)\right|-0}{-h}$

$= \lim\limits_{h\to0^+}\dfrac{(2-h)^2.\left(-\cos\left(\dfrac{\pi}{2-h}\right)\right)-0}{-h} = \lim\limits_{h\to0^+}\dfrac{(2-h)^2\cos\left(\dfrac{\pi}{2-h}\right)}{h}$

$= \lim\limits_{h\to0^+}\dfrac{(2-h)^2.\sin\left(\dfrac{\pi}{2}-\dfrac{\pi}{2-h}\right)}{h} = \lim\limits_{h\to0^+}\dfrac{(2-h)^2.\sin\left(-\dfrac{\pi h}{2(2-h)}\right)}{\left(-\dfrac{\pi h}{2(2-h)}\right)}.\dfrac{-\pi}{2(2-h)} = -\pi$

So f(x) is not derivable at x = 2

8*. $\left.\begin{aligned} f(2n) &= a_n \\ f(2n^+) &= a_n \\ f(2n^-) &= b_n + 1 \end{aligned}\right\}$ $\begin{aligned} a_n &= b_n + 1 \\ a_n - b_n &= 1 \end{aligned}$ So B is correct

$\left.\begin{aligned} f(2n+1) &= a_n \\ f((2n+1)^-) &= a_n \\ f((2n+1)^+) &= b_{n+1} - 1 \end{aligned}\right\}$ $\begin{aligned} a_n &= b_{n+1} - 1 \\ a_n - b_{n+1} &= -1 \text{ So D is correct} \\ a_{n-1} - b_n &= -1 \end{aligned}$

9*. Consider

$h(x) = f(x) - g(x)$ Assume $a < b$

$h(a) = \lambda - g(a) > 0$

$h(b) = f(b) - \lambda < 0$

else if $a > b$ $h(a) < 0$ and $h(b) > 0$.

By intermediate value theorem $\Rightarrow h(c) = 0$ .....(1)

(A) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$

$(f(c) - g(c))\,(f(c) + g(c) + 3) = 0$

So there exist a 'c' : $f(c) - g(c)$

from (1).

Hence A is correct.

(D) Similarly $(f(c))^2 = (g(c))^2$

$(f(c) - g(c))\,(f(c) + g(c)) = 0$

$\Rightarrow$ (D) is correct.

B & C are wrong as by counter eg

If $f(x) = g(x) = \lambda \neq 0$, then

$B \rightarrow \lambda^2 + \lambda = \lambda^2 + 3\lambda$ is not possible.

$C \rightarrow \lambda^2 + 3\lambda = \lambda^2 + \lambda$ is not possible.

10. $\lim\limits_{x\to 1}\left\{\dfrac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \dfrac{1}{4}$

$\Rightarrow \lim\limits_{x\to 1}\left\{\dfrac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{x+\sqrt{x}} = \dfrac{1}{4}$

Hence $\lim\limits_{x\to 1}\left\{\dfrac{-ax+\sin(x-1)+a}{(x-1)+\sin(x-1)}\right\}^{1+\sqrt{x}} = \dfrac{1}{4}$

put $x = 1 + h$,

$\lim\limits_{h\to 0}\left\{\dfrac{-ah+\sin h}{h+\sin h}\right\}^{1+\sqrt{1+h}} = \dfrac{1}{4}$

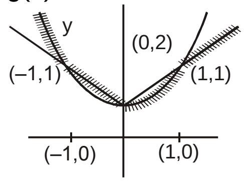
or $\dfrac{-a+1}{2} = \dfrac{1}{2}$ or $-\dfrac{1}{2} \Rightarrow$ $a = 0$ or 2

But at $a = 2$, $\dfrac{-ah+\sin h}{h+\sin h}$ tends to negative value. So correct Answer is $a = 0$

However $a = 2$ may be accepted if this is not considered

11. $f(x) = |x| + 1 = \begin{cases} x+1 & x \ge 0 \\ -x+1 & x < 0 \end{cases}$

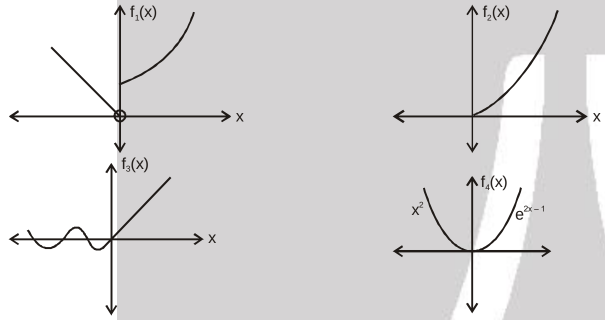
$g(x) = x^2 = 1$



Number of Non-differential points 3.

12. $f_2(f_1(x)) = (f_1(x))^2 - \begin{bmatrix} x^2 & x < 0 \\ e^{2x} & x \ge 0 \end{bmatrix}$

$f_4(x) \begin{bmatrix} x^2 & x < 0 \\ e^{2x} - 1 & x \ge 0 \end{bmatrix}$



$f_4(x)$ is many-one onto, continuous and non-derivable. $f_3(x)$ is many-one, into, continuous and derivable $f_2(x)$ is one-one, into, differentiable. Hence $R \to 2$

so (D)

$p \to 1$, $q \to 3$, $R \to 2$, $S \to 4$

13*. $g(0) = 0$, $\quad g'(0) = 0 \quad g'(1) \ne 0$

$f(x) \begin{cases} g(x) & ; \; x > 0 \\ -g(x) & ' \; x < 0 \\ 0 & ; \; 0 \end{cases} \qquad h(x) = e^{|x|}$

$f(h(x)) = g(e^{|x|})$, $\quad h(f(x)) = e^{|g(x)|}$

$R(f'(0)) = \lim_{x \to 10} \dfrac{g(x) - g(0)}{x - 0} = g'(0) = 0$

$L(f'(0)) = \lim_{x \to 10} = \dfrac{-g(x) - g(0)}{x - 0} \; g'(0) = 0$

$R(h'(0)) = 1$ & $L(h'(0)) = -1$ So h(x) is non derivable. at x = 0

Now $\lim_{x \to 0} \dfrac{f(h(x)) - f(h(0))}{x} = \lim_{x \to 0} \dfrac{g(e^{|x|}) - g(1)}{x}$

$R(f'(h(x))) = \lim_{x \to 0^+} \dfrac{g(e^x) - g(1)}{x} = \lim_{x \to 0^+} \dfrac{g(e^x) - g(1)}{e^x - 1} \dfrac{e^x - 1}{x} = g'(1)$

$L(f'(h(x))) = -g'(1)$ Hence $f(h(x))$ is non derivable at x = 0

Since x = 0 is repeated root of g(x) So $f(h(x))$ is differetiable at x = 0

hence (A), (D)

14*. $f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$

Let $\frac{\pi}{2}$ sin x = θ $\quad\quad \because\ \theta \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\therefore \quad f(x) = \sin\left(\frac{\pi}{6}\sin\theta\right)$

Let $\frac{\pi}{6}\sin\theta = \phi \quad \therefore \quad \phi \in \left[-\frac{\pi}{6},\frac{\pi}{6}\right]$

$\therefore \quad f(x) = \sin\phi \in \left[-\frac{1}{2},\frac{1}{2}\right] \quad\quad \therefore$ **(A)**

Now $fog(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}\sin x\right)\right)\right)$

Clearly, range of fog is also $\left(-\frac{1}{2},\frac{1}{2}\right) \quad\quad \therefore$ **(B)**

Now, $\lim\limits_{x\to 0}\dfrac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{2}\sin x} = \lim\limits_{x\to 0}\dfrac{2}{\pi}\dfrac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}\times\dfrac{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}{\frac{\sin x}{x}\times x}$

$= \lim\limits_{x\to 0}\dfrac{2}{\pi}\times\dfrac{\pi}{6}\times\dfrac{\sin\left(\frac{\pi}{2}\sin x\right)}{\frac{\pi}{2}\sin x}\times\dfrac{\frac{\pi}{2}\sin x}{x} \quad \dfrac{1}{3}\times\dfrac{\pi}{2} = \dfrac{\pi}{6} \quad\quad \therefore$ **(C)**

Now, $\quad gof(x) = 1$

$\Rightarrow \quad \frac{\pi}{2}\sin\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}\sin x\right)\right) = 1$

$\Rightarrow \quad \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right) = \frac{2}{\lambda} \cong \frac{2}{3.14} > \frac{1}{2} \quad\quad \therefore$ **(D)**

15. $m \ge 2$ and $n \ge 2$ = $\lim\limits_{a\to 0}\dfrac{e(e^{\cos(a^n)-1}-1)}{(\cos(a^n)-1)}\times\left(\dfrac{\cos(a^n)-1}{(a^n)^2}\right)\dfrac{a^{2n}}{a^m}$

$= e\times\lim\limits_{a\to 0}\left(\dfrac{e^{\cos(a^n)-1}-1}{\cos(a^n)-1}\right)\times\lim\limits_{a\to 0}\left(\dfrac{\cos(a^n)-1}{a^{2n}}\right)\times\lim\limits_{a\to 0}a^{2n-m} = e\times 1\times -\dfrac{1}{2}\times\lim\limits_{a\to 0}a^{2n-m}$

Now $\lim\limits_{a\to 0}a^{2n-m}$ must be equal to 1. i.e., $2n - m = 0$

$\frac{m}{n} = 2$

16. h(g(g(f(x)) = f(x)
h(g(x)) = f(x) (by definition)
h(g(x) = fof
h(x) = f(fof(x))
g′ (f) = f′ = 1
now $x^2 + 3x + 2 = 2$
we get x = 0
g(f(0)) f′ (0) = 1

$g'(2) = \dfrac{1}{f'(0)} = \dfrac{1}{3} \quad\quad h'(x) = f'(f(x))\,f'(x)$

h′(1) = f′(6) f′(1) = 111 × 6 and h(0) = f(f(0)) = f(2) = 8 + 6 + 2 = 16
and h (g(3)) = f(3) = 27 + 9 + 2 = 38

17. $\lim\limits_{x\to 0}\dfrac{x^2\sin\beta x}{\alpha x-\sin x}=\lim\limits_{x\to 0}\dfrac{x^2\left(\beta x-\dfrac{\beta^3x^3}{3!}+\dfrac{\beta^5x^5}{5!}-.......\right)}{\alpha x-\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-........\right)}=\lim\limits_{x\to 0}\dfrac{x^3\left(\beta-\dfrac{\beta^3x^2}{3!}+.......\right)}{(\alpha-1)x+\dfrac{x^3}{3!}-\dfrac{x^5}{5!}+........}=1$

$\Rightarrow \alpha - 1 = 0 \Rightarrow \alpha = 1$, Limit = $6\beta = 1$

$\Rightarrow\ \beta = \dfrac{1}{6} \Rightarrow 6(\alpha+\beta) = 6\left(1+\dfrac{1}{6}\right) = 6\times\dfrac{7}{6} = 7$

18. $f(x) = [x^2 - 3] = [x^2] - 3 = \begin{cases} -3 & -\dfrac{1}{2}\le x<1 \\ -2 & 1\le x<\sqrt{2} \\ -1 & \sqrt{2}\le x<\sqrt{3} \\ 0 & \sqrt{3}\le x<2 \\ 1 & x=2 \end{cases}$

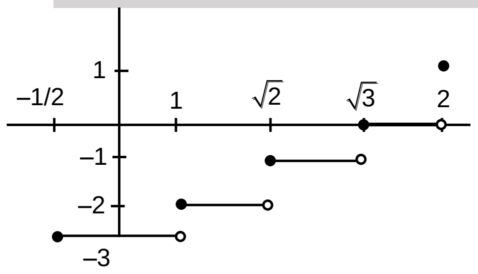
$g(x) = |x|\, f(x) + |4x-7|\, f(x)$

$= (|x| + |4x-7|)\,[x^2-3] = \begin{cases} (-x-4x-7)(-3) & -\dfrac{1}{2}\le x<0 \\ (x-(4x-7))(-3) & 0\le x<1 \\ (x-(4x-7))(-2) & 1\le x<\sqrt{2} \\ (x-(4x-7))(-1) & \sqrt{2}\le x<\sqrt{3} \\ ((x-(4x-7))(0) & \sqrt{3}\le x<7/4 \\ (x+(4x-7))(0) & 7/4\le x<2 \\ (x+(4x-7))(1) & x=2 \end{cases}$

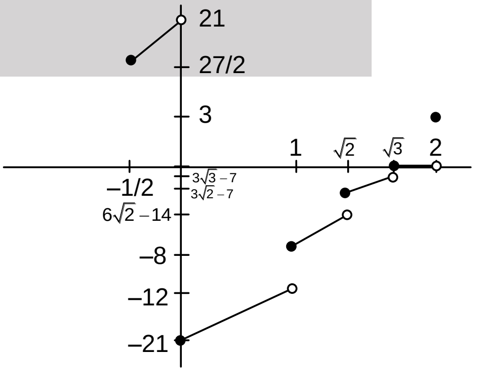
$= \begin{cases} 15x+21 & -\dfrac{1}{2}\le x<0 \\ 9x-21 & 0\le x<1 \\ 6x-14 & 1\le x<\sqrt{2} \\ 3x-7 & \sqrt{2}\le x<\sqrt{3} \\ 0 & \sqrt{3}\le x<2 \\ 5x-7 & x=2 \end{cases}$

Now graph of given function is

f(x)



g(x)



Clearly F is not discontinuous at exactly 4 point in [–1/2, 2] and g is not differentiable at 4 points in (–1/2, 2) Hence Ans. is BC

19. at x = 0, x = 0 is repeated root of $g(x) = |x| \sin|x^3 + x|$ hence f(x) is differentiable

& at x = 1 $\Rightarrow$ $a\cos|x^3 - x| = a\cos(x^3 - x)$

as $\cos(-\theta) = \cos(\theta)$

$\therefore$ f(x) is differentiable

20. $f(x) = x\cos(\pi(x + [x]))$

Check continuity at x = n

$f(n) = n\cos 2n\pi = n$

$f(n^+) = n\cos 2n\pi = n$

$f(n^-) = n\cos(2n-1)\pi = -n$

It is discontinous at all integer points except 0

21*. $f(1^+) = \lim_{h\to 0} \frac{1-(1+h)(1+h)}{h}\cos\frac{1}{h} = \lim_{h\to 0}\frac{1-(1+h)^2}{h}\cos\frac{1}{h} = \lim_{h\to 0}\frac{-h^2-2h}{h}\cos\frac{1}{h}$

$= \lim_{h\to 0}(-h-2)\cos\frac{1}{h} \Rightarrow \lim_{h\to 0} f(1^+)$ does not exist

$f(1^-) = \lim_{h\to 0}\frac{1-(1-h)(1+h)}{h}\cos\frac{1}{h} = \lim_{h\to 0}\frac{1-(1-h^2)}{h}\cos\frac{1}{h} = \lim_{h\to 0}\frac{h^2}{h}\cos\frac{1}{h} = \lim_{h\to 0} h\cos\frac{1}{h} = 0$

22. $f_n(x) = \sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right)$ $f_n(x) = \tan^{-1}(x + n) - \tan^{-1}(x) \Rightarrow f'_n(x) = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$

$f_n(0) = \tan^{-1}(n) \Rightarrow \tan^2(\tan^{-1}n) = n^2$

(A) $\sum_{j=1}^{5}\tan^2(f_j(0)) = \sum_{j=1}^{5} j^2 = \frac{5.6.11}{6} = 55$

(B) $f'_n(0) = \frac{1}{1+n^2} - 1 \Rightarrow 1 + f_n(0) = \frac{1}{1+n^2}$

$\sec^2(f_n(0)) = \sec^2(\tan^{-1}(n)) = 1 + n^2$.

Hence $(1 + f'_n(0)).\sec^2(f_n(0)) = \left(\frac{1}{1+n^2}\right)(1 + n^2) = 1$

so $\sum_{i=1}^{10}(1+f_i'(0))\sec^2(f_i(0)) = \sum_{i=1}^{10} 1 = 10$

$\lim_{x\to\infty} f_n(x) = \lim_{x\to\infty}\tan^{-1}\left(\frac{n}{1+x(n+x)}\right) = 0$

$\lim_{x\to\infty}\tan(f_n(x)) = 0$ & $\lim_{x\to\infty}\sec^2(f_n(x)) = 1$

23. (i) $f'_1(0) = \lim_{h\to 0}\frac{\sin\sqrt{1-e^{-h^2}} - 0}{h} = \lim_{h\to 0}\frac{\sin\sqrt{1-e^{-h^2}}}{\sqrt{1-e^{-h^2}}} \times \sqrt{\frac{1-e^{-h^2}}{h^2}} \times \frac{|h|}{h}$

$= 1 \times 1 \times \frac{|h|}{h} = 1 \times 1 \times \frac{|h|}{h}$ = limit does not exist.

$\Rightarrow$ for option (P), (2) is correct.

(ii) $\lim\limits_{x\to 0} f_2(x) = \lim\limits_{x\to 0}\frac{|\sin x|}{\tan^{-1}x} = \lim\limits_{x\to 0}\frac{|\sin x|}{|x|}\times\frac{x}{\tan^{-1}x}\times\frac{|x|}{x} = \lim\limits_{x\to 0} 1\times 1\times\frac{|x|}{x}$

= limit does not exist $\Rightarrow$ for option Q, (1) is correct.

(iii) $\lim\limits_{x\to 0} f_3(x) = \lim\limits_{x\to 0}[\sin(\log_e(x+2))]$

now at x tends to zero (x + 2) tends to 2

$\Rightarrow$ $\log_e (x + 2)$ tends to $\ell n2$

$\Rightarrow$ $\log_e (x + 2) \to \ell n2$

which is less than 1

$0 < \lim\limits_{x\to 0} \sin(\log_e (x + 2)) < \sin 1$

$\Rightarrow$ $\lim\limits_{x\to 0} [\sin(\log_e(x + 2))] = 0$

$f_3(x) = \{0 \; x \in [-1, e^{\pi/2} - 2)$

$\Rightarrow$ $f'_3(x) = 0$ $\forall x \in (-1, e^{\pi/2} - 2)$

$\Rightarrow$ $f''_3(x) = 0 \; \forall x \in (-1, e^{\pi/2} - 2)$

Hence for (R), (4) is correct.

(iv) $\lim\limits_{x\to 0} f_4(x) = \lim\limits_{x\to 0}\left(x^2\sin\frac{1}{x}\right) = \lim\limits_{x\to 0} x^2\left(\sin\frac{1}{x}\right) = 0$

$f'_4(0) = \lim\limits_{h\to 0}\frac{h^2\sin\left(\frac{1}{x}\right)-0}{x} = \lim\limits_{h\to 0} h\sin\left(\frac{1}{x}\right) = 0$

$f'_4(x) = -\cos\frac{1}{x} + x\sin\frac{1}{x},\; x \neq 0$

$f''_4(0) = \frac{-\cos\frac{1}{h}+h\sin\frac{1}{h}-0}{h} \Rightarrow$ does not exist

hence for (S), (3) is correct.

24. (A) $\lim\limits_{h\to 0}\frac{h^{2/3}-0}{\sqrt{|h|}} = 0$

(B) $\lim\limits_{h\to 0}\frac{\sin h-0}{h^2}$ does not exist

(C) $\lim\limits_{h\to 0}\frac{|h|-0}{\sqrt{|h|}} = 0$

(D) $f(X) = x|x|$

$\Rightarrow$ $\lim\limits_{h\to 0}\frac{f(h)-f(0)}{h^2} = \lim\limits_{h\to 0}\frac{h|h|-0}{h^2}$ does not exist

PART - II

1. $\lim\limits_{x\to 0}\dfrac{f(3x)}{f(x)} = 1$

$f(x) < f(2x) < f(3x)$ Divide by f(x) $1<\dfrac{f(2x)}{f(x)}<\dfrac{f(3x)}{f(x)}$

using sandwitch theorem $\Rightarrow \lim\limits_{x\to\infty}\dfrac{f(2x)}{f(x)} = 1$

Hence correct option is (4)

2. $\lim\limits_{x\to 2}\sqrt{2}\dfrac{|\sin(x-2)|}{(x-2)}$

$\therefore$ does not exist

3. $\lim\limits_{x\to 5}\dfrac{(f(x)^2)-9}{\sqrt{|x-5|}}=0$; $\lim\limits_{x\to 5}[(f(x))^2-9]=0$; $\lim\limits_{x\to 5} f(x) = 3$

4. $f(0) = q$

$f(0^+) = \lim\limits_{x\to 0^+}\dfrac{(1+x)^{1/2}-1}{x} = \lim\limits_{x\to 0^+}\dfrac{1+\frac{1}{2}x+.....-1}{x} = \dfrac{1}{2}$

$f(0^-) = \lim\limits_{x\to 0^-}\dfrac{\sin(p+1)x+\sin x}{x}$

$f(0^-) = \lim\limits_{x\to 0^-}\dfrac{(\cos(p+1)x)(p+1)+(\cos x)}{1} = (p + 1) + 1 = p + 2$

$\therefore$ $p + 2 = q = \dfrac{1}{2} \Rightarrow p = -\dfrac{3}{2}$, $q = \dfrac{1}{2}$

5. $f(x) = \begin{cases} x\sin(1/x) & , \quad x\neq 0 \\ 0 & , \quad x=0\end{cases}$ at x = 0

LHL = $\lim\limits_{h\to 0^+}\left\{-h\sin\left(-\dfrac{1}{h}\right)\right\}$ = 0 × a finite quantity between – 1 and

RHL = $\lim\limits_{h\to 0^+} h\sin\dfrac{1}{h} = 0$

$f(0) = 0$

$\therefore$ f(x) is continuous on R.

$f_2(x)$ is not continuous at x = 0

6. $\lim\limits_{x\to a}\dfrac{x^2f(a)-a^2f(x)}{x-a} = \lim\limits_{x\to a}\dfrac{2xf(a)-a^2f'(x)}{1} = 2af(a) - a^2f'(a)$

Alter $\lim\limits_{x\to a}\dfrac{x^2f(a)-a^2f(x)}{x-a} = \lim\limits_{x\to a}\dfrac{x^2f(a)-a^2f(a)+a^2f(a)-a^2f(x)}{x-a}$

$= \lim\limits_{x\to a}\dfrac{(x^2-a^2)f(a)-a^2(f(x)-f(a))}{x-a} = \lim\limits_{x\to a}(x+a)f(a) - a^2\left\{\dfrac{f(x)-f(a)}{(x-a)}\right\} = 2af(a) - a^2f'(a)$

7. Doubtful points are $x = n$, $n \in I$

L.H.L = $\lim\limits_{x\to n^-}[x]\cos\left(\dfrac{2x-1}{2}\right)\pi = (n-1)\cos\left(\dfrac{2n-1}{2}\right)\pi = 0$

R.H.L. = $\lim\limits_{x\to n^+}[x]\cos\left(\dfrac{2n-1}{2}\right)\pi = n\cos\left(\dfrac{2n-1}{2}\right)\pi = 0$

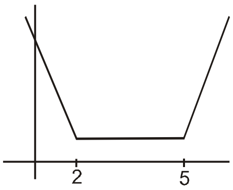
$f(n) = 0$

Hence continuous

8. $f(x) = 3 \qquad 2 \le x \le 5$

$f'(x) = 0 \qquad 2 < x < 5$

$f'(4) = 0$



9. $I = \lim\limits_{x\to 0} \dfrac{(1-\cos 2x)}{x^2}\dfrac{(3+\cos x)}{1}.\dfrac{x}{\tan 4x} = \lim\limits_{x\to 0}\dfrac{2\sin^2 x}{x^2}.\dfrac{3+\cos x}{1}.\dfrac{x}{\tan 4x} = 2.4.\ \dfrac{1}{4} = 2$

10. $\text{Lim}_{x\to 0} \dfrac{\sin(\pi\cos^2 x)}{x^2}\left(\dfrac{0}{0}\text{form}\right) = \text{Lim}_{x\to 0}\dfrac{\sin(\pi\cos^2 x)}{x^2} = \dfrac{\sin(\pi\sin^2 x)}{\pi\sin^2 x} \times \dfrac{\pi\sin^2 x}{x^2} = \pi.$

11. $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x} = \lim\limits_{x\to 0}\dfrac{2\sin^2 x}{x^2}\ \dfrac{(3+\cos x)}{\frac{\sin 4x}{4x}}\ \dfrac{\cos 4x}{4} = 2$

12. $g(x) = \begin{cases} k\sqrt{x+1} & 0 \le x \le 3 \\ mx+2 & 3 < x \le 5 \end{cases}$

$L(g'(3)) = \lim\limits_{x\to 3^-}\dfrac{k\sqrt{x+1}-2k}{x-3} = \lim\limits_{x+3^-} k\left\{\dfrac{(x+1-4)}{(x-3)\left(\sqrt{x+1}+2\right)}\right\} = \dfrac{k}{4}$

$R(g'(3)) = \lim\limits_{x\to 3^+}\dfrac{mx+2-2k}{x-3}$

Since this limit exists 3m + 2 – 2k = 0 $\Rightarrow$ 2k = 3m + 2 ...(i)

So R(g′(3)) = m by L-Hospital rule

Since g(x) is differentiable k = 4m ...(ii)

Solving (i) & (ii)

$m = \dfrac{2}{5}$, $k = \dfrac{8}{5}$ $\Rightarrow$ k + m = 2 ...(ii)

13. $P = \lim\limits_{x\to 0^+}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$ then log p = P = $e^{\lim\limits_{x\to 0^+}\left(1+\tan^2\sqrt{x}-1\right)\frac{1}{2x}} = e^{\lim\limits_{x\to 0^+}\frac{\left(\tan\sqrt{x}\right)^2}{2(\sqrt{x})^2}} = e^{\frac{1}{2}}$

$\log P = \log e^{\frac{1}{2}} = \dfrac{1}{2}$

14. $f(x) = |\ell n2 - \sin x|$

$f(f(x)) = |\ell n2 - \sin|\ell n2 - \sin x||$

In the vicinity of x = 0

g(x) = ℓn2 – sin(ℓn2– sinx)

Hence g(x) is differentiable at x = 0 as it is sum and composite of differentiable function

g'(x) = cos(ℓn2 – sinx). cosx

g'(0) = cos(ℓn2)

15. $x \to \frac{\pi}{2} + t$

$$\lim_{t\to 0} \frac{-\tan t + \sin t}{(2t)^3}$$

$$\lim_{t\to 0} \frac{-\sin t(1-\cos t)}{-8t^3 \cos t} = \frac{1}{16}$$

16. $$\lim_{x\to 0^+} x\left(\left[\frac{1}{x}\right] + \left[\frac{2}{x}\right] + ... + \left[\frac{15}{x}\right]\right)$$

$$\lim_{x\to 0^+} \left(x\left[\frac{1}{x}\right] + x\left[\frac{2}{x}\right] + ... + x\left[\frac{15}{x}\right]\right) = 1 + 2 + 3 + ... + 15 = \frac{15}{2}(15+1) = 120$$

17. f (x) = $|x - \pi|. (e^{|x|} - 1) \sin |x|$ According to given options we have to check only at x = 0 and π

at x = 0, f(0) = 0

LHD = $\lim_{h\to 0^+} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0^+} \frac{(\pi+h).(e^h - 1)\sinh}{-h}$ = RHD = $\lim_{h\to 0^+} \frac{(\pi-h)(e^h-1)\sinh}{h} = 0$

$\Rightarrow$ diff. at x = 0

Now at x = π

f(π) = 0

LHD = $\lim_{h\to 0^+} \frac{f(\pi-h) - f(\pi)}{-h} = \lim_{h\to 0^+} \frac{k.(e^{\pi-h} - 1).\sinh}{h} = 0$

RHD = $\lim_{h\to 0} \frac{f(\pi+h) - f(\pi)}{h} = \lim_{h\to 0} \frac{-h(e^{\pi+h} - 1)\sinh}{h} = 0$

differential at x = π also, hence answer is (3)

18. Using rationalization

$$\lim_{y\to 0} \frac{\sqrt{1+\sqrt{1+y^4}} - \sqrt{2}}{y^4} \times \frac{\sqrt{1+\sqrt{1+y^4}} + \sqrt{2}}{\sqrt{1+\sqrt{1+y^4}} + \sqrt{2}}$$

$$= \lim_{y\to 0} \frac{\sqrt{1+y^4} - \sqrt{1}}{y^4} \times \frac{1}{\sqrt{1+\sqrt{1+y^4}} + \sqrt{2}} \times \frac{\sqrt{1+y^4}+1}{\sqrt{1+y^4}+1} = \lim_{y\to 0} \frac{1}{\sqrt{1+\sqrt{1+y^4}} + \sqrt{2}} \times \frac{1}{\sqrt{1+y^4}+1}$$

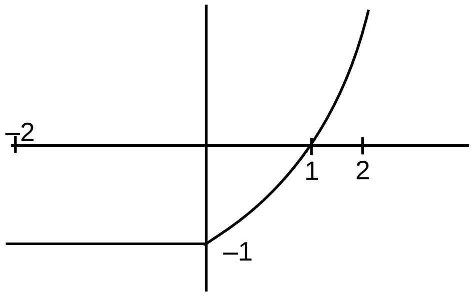
by putting value of limit $= \frac{1}{2\sqrt{2}} \times \frac{1}{2} = \frac{1}{4\sqrt{2}}$

using L.H rule $$\lim_{y\to 0} \frac{\frac{1}{2\sqrt{1+\sqrt{1+y^4}}} \cdot \frac{4y^3}{2\sqrt{1+y^4}}}{4y^3} = \frac{1}{4\sqrt{2}}$$

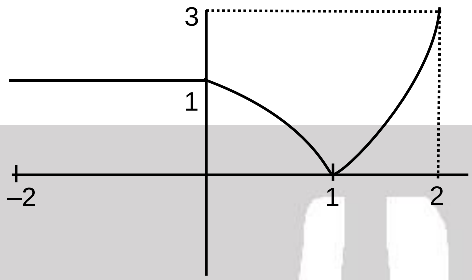
19. $$\lim_{x\to 1^+} \frac{\left(1-|x| + \sin|1-x|\right)\sin\left([1-x]\frac{\pi}{2}\right)}{|1-x|[1-x]}$$

$$= \lim_{x\to 1^+} \frac{\left(1-x+\sin(x-1)\right)\sin\left(-\frac{\pi}{2}\right)}{(x-1)(-1)} = \lim_{x\to 1^+} \frac{-(x-1)+\sin(x-1)}{(x-1)} = -1 + 1 = 0$$

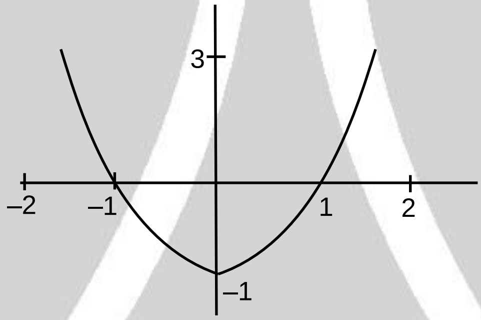
20. y = f(x)



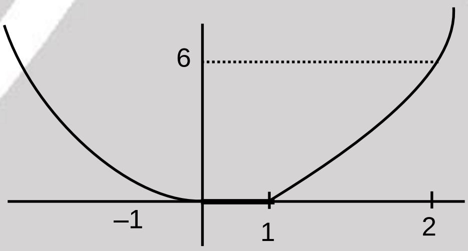
y = |f(x)|



y = f(|x|)



y = g(x)



one non differential point at x = 1

21. Interval f(x) = $\begin{cases} x \in [-1, 0) & \Rightarrow \quad (-x-1) \\ x \in [0, 1) & \Rightarrow \quad x \\ x \in [1, 2) & \Rightarrow \quad 2x \\ x \in [2, 3) & \Rightarrow \quad x+2 \end{cases}$

at x = 0, 1
f(x) is discontinuous

22. $\lim\limits_{x\to 0} \dfrac{(x+2\sin x)\left(\sqrt{x^2+2\sin x+1}+\sqrt{\sin^2 x-x+1}\right)}{x^2+2\sin x+1-\sin^2 x+x-1} = \lim\limits_{x\to 0} \dfrac{\left(1+\left(2\dfrac{\sin x}{x}\right)\right)\left(\sqrt{x^2+2\sin x+1}+\sqrt{\sin^2 x-x+1}\right)}{x+\dfrac{2\sin x}{x}-\dfrac{\sin^2 x}{x}+1}$

$= \dfrac{(1+2)(1+1)}{3} = 2$

23. g(x) = |f(x)|

$g'(c^+) = \lim\limits_{x\to c^+} \dfrac{|f(x)|-f(c)}{x-c} = \lim\limits_{x\to c^+} \dfrac{\pm f(x)}{x-c} = \pm f'(c)$

$g'(c^-) = \lim\limits_{x\to c^-} \dfrac{|f(x)|-f(c)}{x-c} = \lim\limits_{x\to c^-} \dfrac{\pm f(x)}{x-c} = \pm f'(c)$

for g (x) to be differentiable at x = c.

f'(c) must be 0. Else it is non-differentiable.

24. $\because$ f(x) is non differentiable at x = 1, 3, 5

$\Sigma f(f(x)) = f(f(1) + f(f(3)) + f(f(5)) = 1 + 1 + 1 = 3$

25. LHL = a + 3

f(0) = b

RHL = $\lim\limits_{h\to 0}\left(\dfrac{(1+3h)^{\frac{1}{3}}-1}{h}\right)=1$

$\therefore$ a = –2

b = 1

$\therefore$ a + 2b = 0

26. $\lim\limits_{x\to 0} x\left(\dfrac{4}{x}-\left\{\dfrac{4}{x}\right\}\right) = A$

$\Rightarrow \lim\limits_{x\to 0} 4 - x\left\{\dfrac{4}{x}\right\} = A$

$\Rightarrow 4 - 0 = A$

check when

(A) $x = \sqrt{A+1} \Rightarrow x = \sqrt{5} \Rightarrow$ discontinuous

(B) $x = \sqrt{A+21} \Rightarrow x = 5 \Rightarrow$ continuous

(C) $x = \sqrt{A} \Rightarrow x = 2 \Rightarrow$ continuous

(D) $x = \sqrt{A+5} \Rightarrow x = 3 \Rightarrow$ continuous

HIGH LEVEL PROBLEMS (HLP)

1. Let $L_{n+1} = \lim\limits_{x\to 0} \dfrac{1-\cos(a_1x)\ .\ \cos(a_2x).\ \cos(a_3x).......\cos(a_{n+1}x)}{x^2}$

$\Rightarrow L_{n+1} = \lim\limits_{x\to 0} \dfrac{1-\cos(a_{n+1}x) + \cos(a_{n+1}x) - \cos(a_1x)\ .\ \cos(a_2x)\ .....\ \cos(a_{n+1}x)}{x^2}$

$\Rightarrow L_{n+1} = \lim\limits_{x\to 0} \dfrac{1-\cos(a_{n+1}x) + \cos(a_{n+1}x)\ \{1-\cos(a_1x).\cos(a_2x).....\cos(a_nx)\}}{x^2}$

$\Rightarrow L_{n+1} = \lim\limits_{x\to 0} \dfrac{1-\cos(a_{n+1}x)}{x^2} + \lim\limits_{x\to 0} \cos(a_{n+1}x).\ L_n \Rightarrow L_{n+1} = \dfrac{(a_{n+1})^2}{2} + L_n$

$L_1 = \dfrac{a_1^2}{2}$; $L_2 = \dfrac{a_2^2}{2} + L_1 = \dfrac{a_2^2}{2} + \dfrac{a_1^2}{2}$

$L_3 = \dfrac{a_3^2}{2} + L_2 = \dfrac{a_3^2}{2} + \dfrac{a_2^2}{2} + \dfrac{a_1^2}{2}$; $L_n = \dfrac{1}{2}\sum\limits_{i=1}^{n} a_i^2$

2. $f_2(x) = f_1(f_1(x)) = \dfrac{1}{2}\left(\dfrac{x}{2}+10\right)+10$ $f_2(x) = \dfrac{x}{2^2}+\dfrac{10}{2}+10$ $f_n(x) = \dfrac{x}{2^n}+10+\dfrac{10}{2}+\dfrac{10}{2^2}+.....+\dfrac{10}{2^{n-1}}$

$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty}\left(\dfrac{x}{2^n}+10+\dfrac{10}{2}+.....+\infty\right) = 0 + \dfrac{10}{1-\frac{1}{2}} = 20$

3. $\because$ $\lim\limits_{x\to\infty} x^n f(x) = p$ $\Rightarrow$ $\lim\limits_{x\to\infty} \dfrac{x^{n+1}f(x)}{x} = p$

using L- Hospital rule, we get $\lim\limits_{x\to\infty} \dfrac{(n+1)\ x^n f(x) + x^{n+1}f'(x)}{1} = p \Rightarrow \lim\limits_{x\to\infty} x^{n+1} f'(x) = -np.$

4. Let $x_2 = \sqrt{1+1} = \sqrt{2}$; $x_3 = \sqrt{2+1} = \sqrt{3}$; $x_4 = \sqrt{4}$; $x_5 = \sqrt{5}$;......; $x_n = \sqrt{n}$

$\lim\limits_{n\to\infty}\left(\dfrac{x_{n+1}}{x_n}\right)^n = \lim\limits_{n\to\infty}\left(\sqrt{\dfrac{n+1}{n}}\right)^n = e^{\lim\limits_{n\to\infty}\left(\sqrt{\frac{n+1}{n}}-1\right)n} = e^{\lim\limits_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n}} = e^{\lim\limits_{n\to\infty}\left(\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)} = e^{\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{1+\frac{1}{n}}+1}\right)} = e^{\frac{1}{2}} = \sqrt{e}$

5. Now, $\dfrac{1}{n^2+n} < \dfrac{1}{n^2+k} < \dfrac{1}{n^2+1}$ $\Rightarrow$ $\dfrac{k}{n^2+n} < \dfrac{k}{n^2+k} < \dfrac{k}{n^2+1}$

$\Rightarrow$ $\sum\limits_{k=1}^{n}\dfrac{k}{n^2+n} < \sum\limits_{k=1}^{n}\dfrac{k}{n^2+k} < \sum\limits_{k=1}^{n}\dfrac{k}{n^2+1} \Rightarrow$ $\left(\dfrac{1}{n^2+n}\right)\cdot\dfrac{n(n+1)}{2} < \sum\limits_{k=1}^{n}\dfrac{k}{n^2+k} < \dfrac{n(n+1)}{2(n^2+1)}$

$\Rightarrow$ $\lim\limits_{n\to\infty}\left(\dfrac{1}{n^2+n}\right)\cdot\dfrac{n(n+1)}{2} < \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{k}{n^2+k} < \lim\limits_{n\to\infty}\dfrac{n(n+1)}{2(n^2+1)}$

$\Rightarrow$ $\lim\limits_{n\to\infty}\ \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{k}{n^2+k} = \dfrac{1}{2}$ (By Sandwich theorem)

6. $\lim\limits_{x\to\infty} x^3\left\{\sqrt{x^2+\sqrt{1+x^4}} - x\sqrt{2}\right\} = \lim\limits_{x\to\infty} x^3\left[\dfrac{x^2+\sqrt{1+x^4}-2x^2}{\sqrt{x^2+\sqrt{1+x^4}}+x\sqrt{2}}\right]$

$= \lim\limits_{x\to\infty}\dfrac{x^3}{\left(\sqrt{x^2+\sqrt{1+x^4}}+x\sqrt{2}\right)}\ \dfrac{(1+x^4-x^4)}{\left(\sqrt{1+x^4}+x^2\right)} = \lim\limits_{x\to\infty}\dfrac{x^3}{\left(\sqrt{x^2+\sqrt{1+x^4}}+x\sqrt{2}\right)\ \left(\sqrt{1+x^4}+x^2\right)}$

$= \lim\limits_{x\to\infty}\dfrac{x^3}{x^3\left(\sqrt{1+\sqrt{\frac{1}{x^4}+1}} + \sqrt{2}\right)\ \left(\sqrt{1+\frac{1}{x^4}}+1\right)} = \dfrac{1}{2\sqrt{2}}\ .\ \dfrac{1}{2} = \dfrac{1}{4\sqrt{2}}$

7. Let $L = \lim_{x\to\infty} \frac{\log_e(\log_e x)}{e^{\sqrt{x}}} \Rightarrow L = \lim_{x\to\infty} \frac{2\log_e(\log_e x)}{\log_e x} \cdot \frac{\log_e \sqrt{x}}{\sqrt{x}} \cdot \frac{\sqrt{x}}{e^{\sqrt{x}}} = 2 \times 0 \times 0 \times 0 = 0$

Since $\lim_{x\to\infty} \frac{\log_e x}{x} = 0$ and $\lim_{x\to\infty} \frac{x}{e^x} = 0$ (using L.H. Rule)

8. $$\lim_{x\to\frac{\pi}{2}} \frac{\log_e(\sin\ (4m+1)x)}{\log_e(\sin(4n+1)x)}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\sin\ (4m+1)x}.\cos(4m+1)x.(4m+1)}{\frac{1}{\sin\ (4n+1)x}.\cos(4n+1)x.(4n+1)} = \lim_{x\to\frac{\pi}{2}} \frac{\tan\ (4n+1)x.}{\tan(4m+1)x} \cdot \frac{(4m+1)}{4n+1}$$

Put $x-\frac{\pi}{2} = y \quad \Rightarrow \quad x = \frac{\pi}{2} + y \quad \therefore \quad (4m+1)x = (4m+1)\frac{\pi}{2} + (4m+1)y$

$\Rightarrow \quad \tan(4m+1)x = -\cot(4m+1)y$

similarly $\tan(4n+1)x = -\cot(4n+1)y$

$\therefore$ Given limit $= \lim_{y\to 0} \frac{\cot\ (4n+1)y}{\cot\ (4m+1)y} \cdot \frac{(4m+1)}{(4n+1)} = \lim_{y\to 0} \frac{\tan\ (4m+1)y}{\tan\ (4n+1)y} \cdot \frac{(4m+1)}{(4n+1)}$

$$= \lim_{y\to 0} \frac{\tan\ (4m+1)y}{(4m+1)\ y} \cdot \frac{1}{\frac{\tan(4n+1)y}{(4n+1)\ y}} \cdot \frac{(4m+1)^2}{(4n+1)^2} = \frac{(4m+1)^2}{(4n+1)^2}$$

9. Let $\theta = \frac{x}{2^r}$

$f(x) = \tan\theta \,.\, \sec 2\theta = \frac{\sin\theta}{\cos\theta.\cos 2\theta}$

$f(x) = \tan 2\theta - \tan\theta \quad ; \quad f(x) = \sum_{r=1}^{n}\left(\tan\frac{x}{2^{r-1}} - \tan\frac{x}{2^r}\right)$

$f(x) = \tan x - \tan\frac{x}{2n} \quad$ so $f(x) + \tan\frac{x}{2^n} = \tan x$

$$g(x) = \begin{cases} \lim_{n\to\infty} \frac{\log_e \tan x - (\tan x)^n\left[\sin\left(\tan\frac{x}{2}\right)\right]}{1+(\tan x)^n} & x \neq \frac{\pi}{4} \\ k & x = \frac{\pi}{4} \end{cases}$$

$$g(x) = \begin{cases} 0 & \frac{\pi}{4} < x < \frac{\pi}{2} \\ \log_e \tan x & 0 < x < \frac{\pi}{4} \\ k & x = \frac{\pi}{4} \end{cases}$$

It is continuous when k = 0

10. $$\lim_{x\to(e^{-1})^+} \frac{e^{\frac{\ell n(1+\ell nx)}{x}}}{x-e^{-1}} = \lim_{x\to(e^{-1})^+} \frac{e^{\ell n(1+\ell nx)^{1/x}}}{x-e^{-1}} = \lim_{x\to(e^{-1})^+} \frac{(1+\ell nx)^{1/x}}{x-e^{-1}} = \lim_{x\to(e^{-1})^+} \frac{(1+\ell nx).(1+\ell nx)^{\frac{1-x}{x}}}{x-e^{-1}}$$

$$= \lim_{x\to(e^{-1})^+} \frac{e\ell n\ \ ex}{ex-1}.(1+\ell nx)^{\frac{1-x}{x}} = e.1.(0)^{e-1} \left(\because \lim_{x\to 1}\frac{\ell nx}{x-1} = 1\right) = 0$$

11. $P_n = \dfrac{2^3 - 1}{2^3 + 1} \cdot \dfrac{3^3 - 1}{3^3 + 1} \cdot \dfrac{4^3 - 1}{4^3 + 1} \ldots\ldots\ldots\ldots \dfrac{n^3 - 1}{n^3 + 1}$

$\Rightarrow P_n = \dfrac{(2-1)(2^2+2+1)}{(2+1)(2^2-2+1)} \cdot \dfrac{(3-1)(3^2+3+1)}{(3+1)(3^2-3+1)} \ldots\ldots\ldots\ldots \dfrac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)}$

$\Rightarrow P_n = \dfrac{1.7}{3.3} \cdot \dfrac{2.13}{4.7} \cdot \dfrac{3.21}{5.13} \ldots\ldots\ldots \dfrac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)} \Rightarrow P_n = \dfrac{1.2.3\ldots\ldots\ldots\ldots(n-1)}{3.4.5\ldots\ldots\ldots\ldots(n+1)} \cdot \dfrac{7.13.21\ldots\ldots\ldots\ldots(n^2+n+1)}{3.7.13\ldots\ldots\ldots\ldots(n^2-n+1)}$

$\Rightarrow P_n = \dfrac{1.2}{n(n+1)} \cdot \dfrac{n^2+n+1}{3} \Rightarrow P_n = \dfrac{2}{3}\dfrac{(n^2+n+1)}{n(n+1)}$

$\therefore \quad \lim\limits_{n\to\infty} P_n = \lim\limits_{n\to\infty} \dfrac{2}{3}\dfrac{(n^2+n+1)}{n\;(n+1)} = \dfrac{2}{3}$

12. (i) $\lim\limits_{x\to 0}\left(\dfrac{(1+x)^{\frac{1}{x}}}{e}\right)^{\frac{1}{x}} = \lim\limits_{x\to 0}\left(\dfrac{e\left(1-\dfrac{x}{2}+\dfrac{11}{24}x^2-\ldots\ldots\right)}{e}\right)^{\frac{1}{x}} = \lim\limits_{x\to 0}\left(1-\dfrac{x}{2}+\dfrac{11}{24}x^2-\ldots\ldots\right)^{\frac{1}{x}}$

$= e^{\lim\limits_{x\to 0}\left(-\frac{x}{2}+\frac{11}{24}x^2-\ldots\ldots\right)\cdot\frac{1}{x}} = e^{-\frac{1}{2}}$

(ii) $\lim\limits_{x\to 0}\left[\sin^2\left(\dfrac{\pi}{2-ax}\right)\right]^{\sec^2\left(\frac{\pi}{2-bx}\right)}$ (1^∞ form)

$= e^{\lim\limits_{x\to 0}\left(\sin^2\left(\frac{\pi}{2-ax}\right)-1\right)\cdot\sec^2\left(\frac{\pi}{2-bx}\right)} = e^{\lim\limits_{x\to 0}\frac{-\cos^2\left(\frac{\pi}{2-ax}\right)}{\cos^2\left(\frac{\pi}{2-bx}\right)}} \left(\dfrac{0}{0}\text{ form}\right)$

Applying L.Hospital rule $= e^{\lim\limits_{x\to 0}\frac{-\sin\left(\frac{2\pi}{2-ax}\right)}{\sin\left(\frac{2\pi}{2-bx}\right)}\cdot\frac{a\pi\,(2-bx)^2}{b\pi\,(2-ax)^2}} = e^{\lim\limits_{x\to 0}\frac{-\sin\left(\frac{2\pi}{2-ax}\right)}{\sin\left(\frac{2\pi}{2-bx}\right)}\cdot\frac{a\pi}{b\pi}} = e^{\lim\limits_{x\to 0}\frac{-\cos\left(\frac{2\pi}{2-ax}\right)}{\cos\left(\frac{2\pi}{2-bx}\right)}\cdot\frac{\frac{2a^2\pi^2}{(2-ax)^2}}{\frac{2b^2\pi^2}{(2-bx)^2}}} = e^{-\frac{a^2}{b^2}}$

13. $f(x) = \lim\limits_{n\to\infty}\dfrac{x\sin^n x}{\sin^n x+1}$

case (i) if $\sin x \in (0, 1)$

$\lim\limits_{n\to\infty}(\sin^n x) \to 0$

$f(x) = 0$ and $\sin x \in (0, 1) \quad \Rightarrow \quad x \in (2n\pi, 2n\pi + \pi)$

case (ii) if $\sin x = 0 \quad \Rightarrow \quad x \in k\pi,\ k \in z$

$f(x) = 0$

case (iii) if $\sin x = 1 \quad \Rightarrow \quad x \in 2k\pi + \dfrac{\pi}{2},\ k \in z$

$f(x) = \dfrac{x}{2}$

case (iv) if $\sin x \in (-1, 0)$

$\lim\limits_{n\to\infty}(\sin^n x) \to 0$, $x \in (2k\pi + \pi, 2k\pi + 2\pi)$; $k \in z$ $f(x) = 0$

case (v) if $\sin x = -1$

$(\sin^n x) = -1$

then f(x) will not take any definite value so if $\sin x = -1$

$x = \left(2n\pi - \dfrac{\pi}{2}\right)$ these values not lie in the domain

For Domain ; $x \in R - \left\{2n\pi - \dfrac{\pi}{2};\ n \in z\right\}$

For Range ; $f(x) = 0$, for $x \in (2k\pi + \pi, 2k\pi + 2\pi) \cup \{k\pi\} \cup (2k\pi, 2k\pi + \pi)$; $k \in z$

$f(x) = \dfrac{x}{2}$, for $x \in 2k\pi + \dfrac{\pi}{2}$, $k \in z$ Range $= \{0\} \cup \left\{n\pi + \dfrac{\pi}{4};\ n \in z\right\}$

14. $\ell im_{x\to 0}\left(\dfrac{a_1^x+a_2^x}{b_1^x+b_2^x}\right)^{\frac{1}{x}} \equiv e^{\ell im_{x\to 0}\left(\frac{a_1^x+a_2^x}{b_1^x+b_2^x}-1\right)\frac{1}{x}} \equiv e^{\ell im_{x\to 0}\left(\frac{a_1^x-1}{x}+\frac{a_2^x-1}{x}-\frac{(b_1^x-1)}{x}-\frac{(b_2^x-1)}{x}\right)\left(\frac{1}{b_1^x+b_2^x}\right)}$

$= e^{\frac{1}{2}(\log_e a_1+\log_e a_2-\log_e b_1-\log_e b_2)} = e^{\frac{1}{2}\log_e\left(\frac{a_1a_2}{b_1b_2}\right)} \equiv \sqrt{\dfrac{a_1a_2}{b_1\ b_2}}$

15. $\ell im_{x\to 1}\dfrac{p-q+qx^p-px^q}{1-x^p-x^q+x^{p+q}}\left(\dfrac{0}{0}\text{ form}\right) = \ell im_{x\to 1}\dfrac{pqx^{p-1}-pqx^{q-1}}{-px^{p-1}-qx^{q-1}+(p+q)x^{p+q-1}}\left(\dfrac{0}{0}\text{ form}\right)$

$= \ell im_{x\to 1}\dfrac{pq(p-1)x^{p-2}-pq(q-1)x^{q-2}}{-p(p-1)x^{p-2}-q(q-1)x^{q-2}+(p+q)(p+q-1)x^{p+q-2}} = \dfrac{p-q}{2}$

16. $f(n,\theta) = \prod_{r=1}^{n}\left(1-\dfrac{\sin^2\frac{\theta}{2^r}}{\cos^2\frac{\theta}{2^r}}\right)$

$f(n,\theta) = \prod_{r=1}^{n}\left(\dfrac{\cos\frac{\theta}{2^{r-1}}}{\cos\frac{\theta}{2^r}}\right)$; $\quad f(n,\theta) = \dfrac{\cos\theta.\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}..........\cos\frac{\theta}{2^{n-1}}}{\left(\cos\frac{\theta}{2}.\cos\frac{\theta}{2^2}.\cos\frac{\theta}{2^3}.......\cos\frac{\theta}{2^n}\right)^2}$

$f(n,\theta) = \dfrac{\dfrac{\sin 2^n\frac{\theta}{2^{n-1}}}{2^n.\sin\frac{\theta}{2^{n-1}}}}{\left(\dfrac{\sin 2^n.\frac{\theta}{2^n}}{2^n.\sin\frac{\theta}{2^n}}\right)^2}$; $\quad f(n,\theta) = \dfrac{(\sin 2\theta)\left(2^{2n}\sin^2\frac{\theta}{2^n}\right)}{(\sin^2\theta)\left(2^n\sin\frac{\theta}{2^{n-1}}\right)}$

$f(n,\theta) = \ell im_{n\to\infty}\dfrac{\sin 2\theta}{\sin^2\theta}.\left(\dfrac{2^{2n}.\sin^2\frac{\theta}{2^n}}{2^n.\sin\frac{\theta}{2^{n-1}}}\right) = \dfrac{\sin 2\theta}{\sin^2\theta}.\ \dfrac{\theta^2}{2\theta} = \dfrac{\theta}{\tan\theta} = g(\theta) = \ell im_{\theta\to 0}\ g(\theta) = 1$

17. Let $f(x) = \sqrt{\dfrac{4\cos^2 x}{2+\cos x}} \Rightarrow f(\pi) = 2$

$\therefore \lim_{x\to\pi}\dfrac{1}{x-\pi}\left(\sqrt{\dfrac{4\cos^2 x}{2+\cos x}}-2\right) = \lim_{x\to\pi}\dfrac{f(x)-f(\pi)}{x-\pi} = f'(\pi)$

Now $f'(x) = \dfrac{4}{2\sqrt{\dfrac{4\cos^2 x}{2+\cos x}}}\ \dfrac{(2+\cos x)(-2\cos x\sin x)-\cos^2 x(-\sin x)}{(2+\cos x)^2}\quad f'(\pi) = 0$

18. $\ell im_{x\to\infty} x^{a+\frac{1}{3}}\left[\left(1+\dfrac{1}{x}\right)^{1/3}+\left(1-\dfrac{1}{x}\right)^{1/3}-2\right]$; $\ell im_{x\to\infty} x^{a+\frac{1}{3}}\left[1+\dfrac{1}{3x}+\dfrac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{2!\,x^2}+1-\dfrac{1}{3x}+\dfrac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!\,x^2}-2\right]$

$\ell im_{x\to\infty} x^{a+\frac{1}{3}}\left[\dfrac{-2}{9}\ \dfrac{1}{x^2}\right]$ Limit exist when $a+\dfrac{1}{3} = 2 \quad a = \dfrac{5}{3} \quad \lambda = -\dfrac{2}{9}$

$a+\lambda = \dfrac{5}{3}-\dfrac{2}{9} = \dfrac{13}{9}$

19. $f(x) = \lim\limits_{n \to \infty} \dfrac{(1+\sin x)^n + \log x}{2 + (1+\sin x)^n}$

(i) for $0 < \sin x \le 1$,

$$f(x) = \lim_{n\to\infty} \frac{1 + \dfrac{\log x}{(1+\sin x)^n}}{\dfrac{2}{(1+\sin x)^n} + 1} = 1$$

(ii) for $\sin x = 0$,

$$f(x) = \frac{1+\log x}{3}$$

(iii) for $-1 \le \sin x < 0$,

$$f(x) = \frac{\log x}{2}$$

$$\therefore \quad f(x) = \begin{cases} \dfrac{\log x}{2} & , \quad -1 \le \sin x \le 0 \\ \dfrac{1+\log x}{3} & , \quad \sin x = 0 \\ 1 & , \quad 0 < \sin x \le 1 \end{cases}$$

$\therefore$ f(x) is discontinuous at integral multiples of π

20. $\lim\limits_{x \to 0^-} \dfrac{1 - a^x + x\,.\,a^x \ell na}{x^2 a^x} = \lim\limits_{x \to 0^-} \dfrac{-a^x \ell na + \ell na(a^x + x a^x \ \ell na)}{x^2 a^x \ell na + 2x.\ a^x} = \lim\limits_{x \to 0^-} \dfrac{a^x (\ell na)^2}{(x a^x \ell na + 2a^x)} = \dfrac{(\ell na)^2}{2}$

$\lim\limits_{x \to 0^+} \dfrac{(2a)^x - x\ell n\ \ 2a - 1}{x^2} = \lim\limits_{x \to 0^+} \dfrac{(2a)^x \ell n2a \ \ - \ell n2a}{2x} = \lim\limits_{x \to 0^+} \dfrac{(2a)^x (\ell n2a)^2}{2} = \dfrac{(\ell n2a)^2}{2}$

for g(x) to be continuous $(\ell na)^2 = (\ell n2a)^2$

$\Rightarrow \quad (\ell na + \ell n2a) = 0 \qquad \Rightarrow \qquad a = \dfrac{1}{\sqrt{2}}$

$\therefore \quad g(0) = \dfrac{1}{8}\ (\ell n\ 2)^2$

21. Let $\theta = \sin^{-1} x \quad \because$ as $\quad x \to \dfrac{1}{\sqrt{2}} \qquad \therefore\ \theta \to \dfrac{\pi}{4}$

$$\lim_{\theta\to\frac{\pi}{4}} \frac{\cos^{-1}(2\sin\theta\cos\theta)}{\sin\theta - \dfrac{1}{\sqrt{2}}} = \lim_{\theta\to\frac{\pi}{4}} \frac{\cos^{-1}(\sin 2\theta)}{\sin\theta - \dfrac{1}{\sqrt{2}}} = \lim_{\theta\to\frac{\pi}{4}} \frac{\cos^{-1}\left(\cos\left(\dfrac{\pi}{2} - 2\theta\right)\right)}{\sin\theta - \dfrac{1}{\sqrt{2}}}$$

Left hand limit $= \lim\limits_{\theta\to\frac{\pi}{4}^-} \dfrac{\cos^{-1}\left(\cos\left(\dfrac{\pi}{2} - 2\theta\right)\right)}{\sin\theta - \dfrac{1}{\sqrt{2}}} = \lim\limits_{\theta\to\frac{\pi}{4}^-} \dfrac{\dfrac{\pi}{2} - 2\theta}{\sin\theta - \dfrac{1}{\sqrt{2}}} \left(\dfrac{0}{0} \text{ form}\right) = \lim\limits_{\theta\to\frac{\pi}{4}^-} \dfrac{-2}{\cos\theta} = -2\sqrt{2}$

Right hand limit $= \lim\limits_{\theta\to\frac{\pi}{4}^+} \dfrac{\cos^{-1}\left(\cos\left(\dfrac{\pi}{2} - 2\theta\right)\right)}{\sin\theta - \dfrac{1}{\sqrt{2}}} = \lim\limits_{\theta\to\frac{\pi}{4}^+} \dfrac{\cos\left(\cos\left(2\theta - \dfrac{\pi}{2}\right)\right)}{\sin\theta - \dfrac{1}{\sqrt{2}}} = \lim\limits_{\theta\to\frac{\pi}{4}^+} \dfrac{2\theta - \dfrac{\pi}{2}}{\sin\theta - \dfrac{1}{\sqrt{2}}}$

$= \lim\limits_{\theta\to\frac{\pi}{4}^+} \dfrac{2}{\cos\theta} = 2\sqrt{2} \Rightarrow$ LHL $\ne$ RHL

$\therefore$ Limit does not exist

22. $\because$ R.H.L. = $\lim_{h \to 0^+} f(0 + h) = \dfrac{\cos^{-1}(1-\{h\}^2)\sin^{-1}(1-\{h\})}{\{h\}-\{h\}^3} = \lim_{h \to 0^+} \dfrac{\cos^{-1}(1-h^2)}{h} \cdot \lim_{h \to 0^+} \dfrac{\sin^{-1}(1-h)}{1-h^2}$

(putting $1-h^2 = \cos 2\theta$)

$= (\sin^{-1}1) \lim_{\theta \to 0^+} \dfrac{\cos^{-1}(1-2\sin^2\theta)}{\sqrt{2}\sin\theta} = \dfrac{\pi}{2\sqrt{2}} \lim_{\theta \to 0^+} \dfrac{2\theta}{\sin\theta} = \dfrac{\pi}{\sqrt{2}}$

$\because$ L.H.L = $\lim_{h \to 0^+} f(0 - h) = \lim_{h \to 0^+} \dfrac{\cos^{-1}(1-\{-h\}^2)\sin^{-1}(1-\{-h\})}{\{-h\}-\{-h\}^3} = \lim_{h \to 0^+} \dfrac{\cos^{-1}(h(2-h))\sin^{-1}h}{(1-h)(2-h)h}$

$= \lim_{h \to 0^+} \dfrac{\cos^{-1}(h(2-h))}{(1-h)\ (2-h)} \lim_{h \to 0^+} \dfrac{\sin^{-1}h}{h} = \dfrac{\cos^{-1}0}{2} = \dfrac{\pi}{4}$

since R.H.L. $\neq$ L.H.L, therefore no value of f(0) can make f continuous at x = 0

23. As f is continuous on R, so $f(0) = \underset{x \to 0}{\text{limit}}\ f(x)$

Thus $f(0) = \underset{n \to \infty}{\text{limit}}\ f\left(\dfrac{1}{4n}\right) = \underset{n \to \infty}{\text{limit}} \left((\sin e^n)\ e^{-n^2} + \dfrac{1}{1 + \dfrac{1}{n^2}} \right) = 0 + 1 = 1$

24. f(0) = 0

$f(0^+) = \lim_{h \to 0} \dfrac{h}{h+1} + \dfrac{h}{(h+1)(2h+1)} + \dfrac{h}{(2h+1)(3h+1)} + \ldots\ldots\ldots\ldots\infty$

$= \lim_{h \to 0} \left\{ 1 - \dfrac{1}{h+1} + \dfrac{1}{(h+1)} - \dfrac{1}{2h+1} + \dfrac{1}{2h+1} - \dfrac{1}{3h+1} + \ldots\ldots\ldots\infty \right\}$

f(x) is not continous at x = 0

since $f(0) \neq f(0^+)$

25. Let F(x) = f(x) – x

F(a) = f(a) – a = b – a

F(b) = f (b) – b = a – b

$\because$ F(x) is continous and F (a) & F (b) are of opposite signs.

$\therefore$ F(x) = 0 has atleast one root in (a, b)

f (x) – x = 0

$\Rightarrow$ f (x) = x for atleast one c $\in$ (a, b)

26. f(x.y) = f(x). f(y)

Put x = y = 1 f(1) = $f^2(1)$ f(1) = 1 since f(1) $\neq$ 0

Now let x $\neq$ 0

then $\lim_{h \to 0} f(x+h) = \lim_{h \to 0} f(x(1+h/x)) = \lim_{h \to 0} f(x)\ \ f\ \ (1+h/x))$

$= f(x).f(1) = f(x)$

also $\lim_{h \to 0} f\ \ (x-h) = \lim_{h \to 0} f\ \ (x)f(1-h/x) = f(x).f(1) = f(x)$

$\Rightarrow$ f(x) is continous for all x except at x = 0

27. $g(x) = \begin{cases} \dfrac{f(x)}{2} & x > 1 \\ \dfrac{h(x)+1}{6} & x < 1 \end{cases}$

$g(1) = \lim\limits_{x\to 1} \dfrac{\sin^2 \pi \ . \ 2^x}{\log_e \sec \pi \ . \ 2^x} = \lim\limits_{x\to 1} \dfrac{\sin^2(2\pi - \pi \ . \ 2^x)}{\log_e \sec(2\pi - \pi \ . \ 2^x)} = 2$

Now g(x) is continuous at x = 1

$g(1^+) = g(1^-) = g(1)$

$\dfrac{f(1)}{2} = \dfrac{h(1)+1}{6} = 2$

f(1) = 4 h(1) = 11

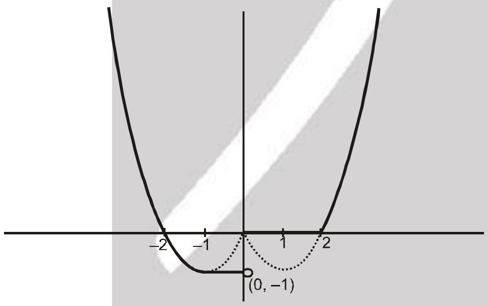
g(1) = 2

4g(1) + 2f(1) – h(1) = 8 + 8 – 11 = 5

28. $\because$ $f(x) = \begin{cases} x^2 + 2x & , \ x < 0 \\ x^2 - 2x & , \ x \geq 0 \end{cases}$

by definition of g(x)

$g(x) = \begin{cases} x^2 + 2x & , \ -2 \le x < -1 \\ -1 & , \ -1 \le x < 0 \\ 0 & , \ 0 \le x < 2 \\ x^2 - 2x & , \ 2 \le x < 3 \end{cases}$

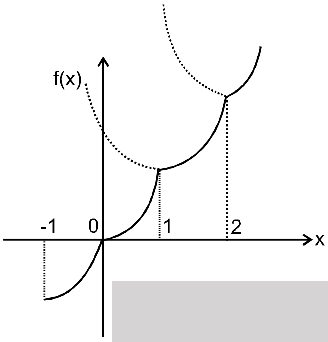


$g'(x) = \begin{cases} 2x + 2 & , \ -2 \le x < -1 \\ 0 & , \ -1 \le x < 0 \\ 0 & , \ 0 \le x < 2 \\ 2x - 2 & , \ 2 \le x < 3 \end{cases}$

clearly g(x) is discontinuous at x = 0 and not differentiable at x = 0, 2

29. $f(x) = \begin{cases} -1+(x+1)^2 & , & -1 \le x < 0 \\ x^2 & , & 0 \le x < 1 \\ 1+(x-1)^2 & , & 1 \le x < 2 \\ 2+(x-2)^2 & , & 2 \le x < 3 \end{cases}$

graph of f(x) is as shown is figure



$\therefore$ f(x) is continuous for all x but

non-differentiable for integral points.

30. $\because$ $f(x) = \begin{cases} \dfrac{x}{1-x} & , & x \le -1 \\ \dfrac{x}{1+x} & , & -1 < x < 0 \\ \dfrac{x}{1-x} & , & 0 \le x < 1 \\ \dfrac{x}{1+x} & , & x \ge 1 \end{cases}$

$\because$ $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{x}{1-x} = 0$ and $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{x}{1+x} = 0$

and $f(0) = 0$ $\therefore$ f(x) is continuous at x = 0

$\because$ $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \dfrac{x}{1+x} = \dfrac{1}{2}$ and $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} \dfrac{x}{1-x} \to \infty$

$\therefore$ f(x) is discontinuous for x = 1

Similarly we can check that f(x) is discontinuous at x = – 1

$\because$ L.H.D. at (x = 0) is = $\lim\limits_{h\to 0^+} \dfrac{f(0-h)-f(0)}{-h} = \lim\limits_{h\to 0^+} \dfrac{\frac{-h}{1+h}-0}{-h} = 1$

$\because$ R.H.D. at (x = 0) is = $\lim\limits_{h\to 0^+} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0^+} \dfrac{\frac{h}{1-h}-0}{h} = 1$

$\Rightarrow$ L.H.D. = R.H.D.

$\therefore$ at x = 0, f(x) is derivable.

31. $\left(\lim\limits_{m \to \infty} \cos^{2n}(m!\ \ \pi x)\right)$ is 1 or 0 according to x is a rational number or an irrational number. As m!πx will become integral multiple of π when x is rational, then cos (m!πx) = ± 1. And when m!πx is not an integral multiple of π i.e. when x is irrational then –1 < cos (m!πx) < 1

$\therefore$ f(x) = $\begin{cases} 0 \quad , & x \in Q \\ -1 \quad , & x \notin Q \end{cases}$

$\therefore$ f(x) is discontinuous and non-differentiable at every real number.

32. $\because$ f(1) = 0

R.H.L. = $\lim\limits_{h \to 0^+}$ f(1 + h) = $\lim\limits_{h \to 0^+} \cos^{-1}\left(\text{sgn}\left(\frac{2}{3+3h-1}\right)\right) = \lim\limits_{h \to 0^+} \cos^{-1}\left(\text{sgn}\left(\frac{2}{2+3h}\right)\right) = 0$

L.H.L. = $\lim\limits_{h \to 0^+}$ f(1 – h) = $\lim\limits_{h \to 0^+} \cos^{-1}\left(\text{sgn}\left(\frac{0}{3-3h}\right)\right) = \cos^{-1}(0) = \frac{\pi}{2}$

$\because$ f(–1) = 0

$\therefore$ f(x) is discontinuous hence non-derivable at x = 1

$\because$ $f'(-1^+) = \lim\limits_{h \to 0^+} \frac{f(-1+h)-f(-1)}{h} = 0$ and $f'(-1^-) = \lim\limits_{h \to 0^+} \frac{f(-1-h)-f(-1)}{h} = 0$

$\Rightarrow$ $f'(-1^+) = f'(-1^-) = 0$

$\therefore$ f(x) is derivable at x = – 1

33. y = f(x) = xsin 1/x. sin $\frac{1}{x\sin 1/x}$ when $x \neq 0, \frac{1}{r\pi}$, r = 1,2,3

y = 0, x = 0, $\frac{1}{r\pi}$ where r = 1, 2, 3,..............

Let t = x sin1/x as x $\to 0^+$, t $\to 0$ and as x $\to \frac{1}{r\pi}$, t $\to$ 0

y = t sin1/t

$\lim\limits_{x \to 0} y = \lim\limits_{t \to 0} t\sin t = 0$ = f(0) also $\lim\limits_{x \to \frac{1}{r\pi}} y = \lim\limits_{t \to 0} t\sin t = 0$ = f $\left(\frac{1}{r\pi}\right)$

f(x) is continous at x = 0 and $\frac{1}{r\pi} \Rightarrow$ f(x) is continous $\forall$ $x \in [0,\ \ 1]$

We know that t = xsin1/x is not differentiable at x = 0

therefore y = tsin1/t = xsin1/x. sin $\frac{1}{x\sin\frac{1}{x}}$ is not differentiable at x = 0

34. $\because$ $f'(x) = \underset{h \to 0}{\text{limit}} \frac{f(h+x)-f(x)}{h} = \underset{h \to 0}{\text{limit}} \frac{f(x)f\left(1+\frac{h}{x}\right)-f(x)}{h} = f(x) \underset{h \to 0}{\text{limit}} \frac{\frac{h}{x}\left(1+g\left(\frac{h}{x}\right)\right)}{h}$

$= \frac{f(x)}{x}\left(\underset{h \to 0}{\text{limit}}\left(1+g\left(\frac{h}{x}\right)\right)\right) \Rightarrow f'(x) = \frac{f(x)}{x}$ so $\int \frac{f(x)}{f'(x)}\,dx = \int x\,dx = \frac{x^2}{2}+c$

35. Given that

$f(xy) = e^{xy-x-y}\,(e^y f(x) + e^x f(y)) \quad \forall\ x,y \in R^+$

putting x = y = 1 , we get

$f(1) = e^{-1}\,(ef(1) + ef(1)) \Rightarrow f(1) = 0$

now $\quad \because \quad f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{e^{x\left(1+\frac{h}{x}\right)-x-\left(1+\frac{h}{x}\right)}\left\{e^{1+\frac{h}{x}}f(x)+e^x f\left(1+\frac{h}{x}\right)\right\}-f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{f(x)(e^h-1)+e^{h-1-\frac{h}{x}+x}\left(f\left(1+\frac{h}{x}\right)-f(1)\right)}{h} = f(x) + \dfrac{e^{x-1}\ .\ f'(1)}{x}$

$\Rightarrow \quad f'(x) = f(x) + \dfrac{e^x}{x} \Rightarrow \dfrac{1}{x} = \dfrac{e^x f'(x) - f(x)e^x}{e^{2x}} \Rightarrow \dfrac{1}{x} = \dfrac{d}{dx}\left(\dfrac{f(x)}{e^x}\right)$

Integrating both sides w.r.t. 'x', we get

$\ell n\,|x| + c = \dfrac{f(x)}{e^x} \quad$ or $\quad f(x) = e^x\,(\ell n|x| + c)$

since $f(1) = 0 \Rightarrow c = 0$

$\therefore\ f(x) = e^x \ell n|x|$

36. Given $f(x + y^3) = f(x) + [f(y)]^3$ and $f'(0) \geq 0$

putting x = y = 0, we get $f(0) = f(0) + (f(0))^3 \Rightarrow f(0) = 0$

also $f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{f(h)}{h}$

Let $L = f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+(h^{1/3})^3)-f(0)}{(h^{1/3})^3} = \lim\limits_{h\to 0} \dfrac{\left(f(h^{1/3})\right)^3}{(h^{1/3})^3} = L^3$

or $L = L^3$ or $L = 0, 1, -1$ as $f'(0) \geq 0 \Rightarrow f'(0) = 0, 1$

Thus $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f\left(x+(h^{1/3})^3\right)-f(x)}{(h^{1/3})^3}$

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(x)+(f(h^{1/3}))^3-f(x)}{(h^{1/3})^3} \Rightarrow f'(x) = 0 , 1$

Integrating both sides, we get f(x) = 0 or f(x) = x + c

As f(0) = 0 , we have f(x) = 0 or f(x) = x

Now f(x) = 0 is imposible as f(x) is not identically zero

$\therefore \quad f(x) = x$ and $\quad f(10) = 10$

37. $f(x) + f\left(\dfrac{1}{1-x}\right) = \dfrac{2(2-2x)}{x(1-x)} = g(x) \quad$ ....(i)

Replace x by $\dfrac{1}{1-x}$ in equation (i)

$f\left(\dfrac{1}{1-x}\right) + f\left(\dfrac{1-x}{-x}\right) = g\left(\dfrac{1}{1-x}\right) \quad$ ....(ii)

Replace x by $\dfrac{1-x}{-x}$ in equation (i)

$f\left(\dfrac{1-x}{-x}\right) + f(x) = g\left(\dfrac{1-x}{-x}\right) \quad$ ....(iii)

(i) – (ii) + (iii)

$2f(x) = g(x) - g\left(\dfrac{1}{1-x}\right) + g\left(\dfrac{1-x}{-x}\right) = \dfrac{2(1-2x)}{x(1-x)} - \dfrac{2(1-x^2)}{x} + \dfrac{2x(2-x)}{(x-1)} = \dfrac{2(x+1)}{x-1}$ or $f(x) = \dfrac{x+1}{x-1}$

38. Let x = y = 1

f(x) + f(y) + f(xy) = 2 + f(x) . f(y)

3f (1) = 2 + $(f(1))^2$ $\Rightarrow$ f(1) = 1, 2. But given that

f(1) $\neq$ 1 so f(1) = 2

Now put y = $\frac{1}{x}$

$f(x) + f\left(\frac{1}{x}\right) + f(1) = 2 + f(x) . f\left(\frac{1}{x}\right) \Rightarrow f(x) + f\left(\frac{1}{x}\right) = f(x) . f\left(\frac{1}{x}\right)$

so f(x) = $\pm x^n + 1$

Now f(4) = 17 $\Rightarrow \pm (4)^n + 1 = 17 \Rightarrow n = 2$

f(x) = $+(x)^2 + 1$.

$\therefore$ f(5) = $5^2 + 1 = 26$

39. $| f(2^k) - f(2^i)| = | f(2^k) - f(2^{k-1}) + f(2^{k-1}) - f(2^{k-2}) .... f(2^{i+1}) - f(2^i)|$

$\leq | f(2^k) - f(2^{k-1})| + | f(2^{k-1}) - f(2^{k-2})| + ...| f(2^{i+1}) - f(2^i)|$

Consider $| f(2^{k-1} + 2^{k-1})| - f(2^{k-1}) | \leq 1$

So $| f(2^k) - f(2^i)| \leq 1 + 1 + ...(k - i)$ term $\leq$

$\sum | f(2^k) - f(2^i) | \leq \sum_{i=1}^{k} (k - i) \leq \frac{k\ (k-1)}{2}$ Hence proved.

Hindi. $| f(2^k) - f(2^i)| = | f(2^k) - f(2^{k-1}) + f(2^{k-1}) - f(2^{k-2}) .... f(2^{i+1}) - f(2^i)|$

$\leq | f(2^k) - f(2^{k-1})| + | f(2^{k-1}) - f(2^{k-2})| + ...| f(2^{i+1}) - f(2^i)|$

$| f(2^{k-1} + 2^{k-1})| - f(2^{k-1}) | \leq 1$

$| f(2^k) - f(2^i)| \leq 1 + 1 + ...(k - i)$

$\sum | f(2^k) - f(2^i) | \leq \sum_{i=1}^{k} (k - i) \leq \frac{k\ (k-1)}{2}$

40. Notice that f(f(x)) – f(x) = x and if f(x) = f(y) then clearly x = y. This means that the function is injective. Since f(f(0)) = f(0) + 0 = f(0), because of injectivity we must have f(0) = 0, implying f(f(0)) = 0. If there were another x such that f(f(x)) = 0 = f(f(0)), injectivity would imply f(x) = f(0) and x = 0.

41. Let xy = u and x/y = v $\Rightarrow$ $2f\left(\sqrt{uv}\right) = f(u) + f(v)$

differentiating w.r.t. u, we get $2f'\left(\sqrt{uv}\right)\frac{\sqrt{v}}{2\sqrt{u}} = f'(u)$

put u = 1 and replace v with t^2 $\Rightarrow$ $2f'(t).\frac{t}{2} = f'(1)$ $\Rightarrow$ $f'(t) = \frac{1}{t}$

$\Rightarrow$ f(t) = log t + c

as $f'(1) = 1$ $\Rightarrow$ c = 0 $\Rightarrow$ f(t) = log t

42. (i) Put x = y = 1 in given relation, we get f(f(1)) = f(1)

(ii) Now put x = 1, y = f(1) in given relation, we get $f(f(1)) = \frac{f(1)}{f(1)} = 1$

from (i) and (ii)

$\therefore$ f(f(1)) = 1

$\therefore$ f(1) = 1

Put x = 1, $f(f(y)) = \frac{f(1)}{y} \Rightarrow f(f(y)) = \frac{1}{y}$ now substitute y = f(x)

$f(f(f(x))) = \frac{1}{f(x)} \Rightarrow f\left(\frac{1}{x}\right) = \frac{1}{f(x)}$

43. **(i)** $f(x + p) = \{1 + (1 - f(x))^3\}^{1/3}$

$\Rightarrow \quad f(x + p) + f(x) = 2$ ....(i)

replacing x by (x + p) in equation (i)

$f(x + 2p) + f(x + p) = 2$ ......(ii)

from (i) and (ii)

$f(x) = f(x + 2p) \Rightarrow$ period = 2p

(ii) $f(x - 1) + f(x + 3) = f(x + 1) + f(x + 5)$ .....(i)

Replace x by x + 2

$f(x + 1) + f(x + 5) = f(x + 3) + f(x + 7)$ ......(ii)

(i) – (ii)

$f(x - 1) = f(x + 7)$

$\Rightarrow f(x + 8) = f(x)$

period = 8

44. Number of ways of discontinuity at a point x = c are 3.

(i) LHL at x = c is equal to f(c) but not equal to RHL at x = c

(ii) LHL at x = c is not equal to f(c) but equal to RHL at x = c

(iii) LHL at x= c is neither equal to f(c) nor equal to RHL at x = c (where f(c)) is equal to RHL at x=c

45. **(i)** $h(x) = \begin{cases} \alpha & , \quad x \in (-\infty, \beta) \\ \beta & , \quad x = \beta \\ \gamma & , \quad x \in (\beta, \infty) \end{cases}$

(ii) $h(x) = \begin{cases} \alpha & , \quad x \in (-\infty, \alpha] \\ \beta & , \quad (\alpha, \gamma) \\ \gamma & , \quad [\gamma, \infty) \end{cases}$